Central Limit Theorem for Linear Eigenvalue Statistics for Submatrices of Wigner Random Matrices

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Abstract

We prove the Central Limit Theorem for finite-dimensional vectors of linear eigenvalue statistics of submatrices of Wigner random matrices under the assumption that test functions are sufficiently smooth. We connect the asymptotic covariance to a family of correlated Gaussian Free Fields.

1 Introduction

The goal of this paper is to prove the central limit theorem for the joint distribution of linear eigenvalue statistics for submatrices of Wigner random matrices.

Let $\{W_{jj}\}_{j=1}^n$ and $\{W_{jk}\}_{1\le j<k\le n}$ be two independent families of independent and identically distributed real-valued random variables satisfying:

$$\mathbb{E}[W_{jk}]=0, \quad \mathbb{E}[W_{jk}^2]=1 \quad \text{for} \quad j<k, \quad \text{and} \quad \mathbb{E}[W_{jj}^2]=\sigma^2. \tag{1.1}$$

Set $W=(W_{jk})_{j,k=1}^n$ with $W_{jk}=W_{kj}$. The Wigner Ensemble of normalized real symmetric $n\times n$ matrices consists of matrices M of the form

$$M=\frac{1}{\sqrt{n}}W. \tag{1.2}$$

For a real symmetric (Hermitian) matrix M of order n, its empirical distribution of the eigenvalues is defined as $\mu_M=\frac{1}{n}\sum_{i=1}^n\delta_{\lambda_i}$, where $\lambda_1\le\ldots\le\lambda_n$ are the (ordered) eigenvalues of M. The Wigner semicircle law states that for any bounded continuous test function $\varphi:\mathbb{R}\to\mathbb{R}$, the linear statistic

$$\frac{1}{n}\sum_{i=1}^n\varphi(\lambda_i)=\frac{1}{n}\mathrm{Tr}(\varphi(M))=:\mathrm{tr}_n(\varphi(M)) \tag{1.3}$$

converges to $\int\varphi(x)d\mu_{sc}(dx)$ in probability, where μ_{sc} is determined by its density

$$\frac{d\mu_{sc}}{dx}(x)=\frac{1}{2\pi}\sqrt{4-x^2}\mathbf{1}_{[-2,2]}(x), \tag{1.4}$$

see e.g. [13], [5], [1].

2 Statement of Main Results

For a generic random variable ξ, in what follows denote by $\xi^\circ := \xi - \mathbb{E}[\xi]$. For a finite set $B \subset \{1, 2, \dots, n\}$ denote by $M(B)$ the submatrix of M formed by the entries corresponding to intersections of rows and columns of M marked by the indices in B, which inherits the ordering. For example,

$$M(\{1,3\}) = \begin{pmatrix} M_{11} & M_{13} \\ M_{31} & M_{33} \end{pmatrix}. \tag{2.1}$$

Let $\mathcal{B}_1, \cdots, \mathcal{B}_d$ be infinite subsets of $\mathbb{N}$ such that $\mathcal{B}_i,\ 1 \le i \le d$, and their pairwise intersections have positive densities. Denote

$$B_i^n = \mathcal{B}_i \cap \{1, 2, \dots, n\},\ 1 \le i \le d, \tag{2.2}$$

$$n_i = |B_i^n|,\ 1 \le i \le d, \tag{2.3}$$

$$n_{lm} = |B_l^n \cap B_m^n|,\ 1 \le l \le m \le d. \tag{2.4}$$

We assume that the following limits exist:

$$\gamma_l := \lim_{n\to\infty} \frac{n_l}{n} > 0, \quad \gamma_{lm} := \lim_{n\to\infty} \frac{n_{lm}}{n}, \quad 1 \le l \le m \le d. \tag{2.5}$$

If it does not lead to ambiguity, we will omit the superindex n in the notation for $B_i^n,\ 1 \le i \le d$.

For an $n \times n$ matrix M and $B \subset \{1, 2, \dots, n\}$, consider a spectral linear statistic $\sum_{l=1}^{|B|} \varphi(\lambda_l)$, where $\{\lambda_l\}_{l=1}^{|B|}$ are the eigenvalues of the submatrix $M(B)$. We are going to study the joint fluctuations of linear statistics of the eigenvalues. It will be beneficial later to view the submatrices from a different perspective. Consider the matrix $P^B = \mathrm{diag}(P_{jj}^B)$, which projects onto the subspace corresponding to indices in B, i.e.

$$P_{jj}^B = \mathbf{1}_{\{j \in B\}},\ 1 \le j \le n. \tag{2.6}$$

Define

$$M^B := P^B M P^B, \tag{2.7}$$

$$\mathcal{N}_B[\varphi] := \sum_{l=1}^{n} \varphi(\lambda_l^B) = Tr(\varphi(M^B)), \tag{2.8}$$

where $\{\lambda_l^B\}_{l=1}^n$ are the eigenvalues of M^B. Note that the spectra of M^B and $M(B)$ differ only by a zero eigenvalue of multiplicity $n - |B|$. As a result, when we consider the linear statistics of their eigenvalues the extra terms $(n - |B|)\varphi(0)$ cancel when we center these random variables. In general, when considering multiple sequences B_l, to simplify the notation write

$$M^{(l)} := M^{B_l}, \quad P^{(l)} := P^{B_l}, \quad \mathcal{N}_n^{(l)}[\varphi] := \mathcal{N}_{B_l}[\varphi], \quad \mathcal{N}_n^{(l)\circ}[\varphi] = \mathcal{N}_n^{(l)}[\varphi] - \mathbb{E}\{\mathcal{N}_n^{(l)}[\varphi]\}. \tag{2.9}$$

Also, denote by $P^{(l,r)}$ the matrix which projects onto the subspace corresponding to the indices in the intersection $B_l \cap B_r$, i.e.

$$P^{(l,r)} = P^{(l)} P^{(r)}. \tag{2.10}$$

Recall that a test function φ belongs to the Sobolev space $\mathcal{H}_s$ if

$$||\varphi||_s^2 := \int_{-\infty}^{\infty} (1 + |t|)^{2s} |\widehat{\varphi}(t)|^2 dt \ < \ \infty. \tag{2.11}$$

First we consider Gaussian Wigner matrices.

Theorem 2.1. *Let $W = \{W_{jk} : W_{jk} = W_{kj}\}_{j,k=1}^{n}$ be an $n \times n$ real symmetric random matrix with Gaussian entries satisfying (1.1) and $M = n^{-1/2}W$. Let $\mathcal{B}_1, \ldots \mathcal{B}_d$ be infinite subsets of $\mathbb{N}$ satisfying (2.2-2.5). Let $\varphi_1, \cdots, \varphi_d : \mathbb{R} \to \mathbb{R}$ be test functions that satisfy the regularity condition $||\varphi_l||_s < \infty$, for some $s > \frac{5}{2}$. Then the random vector*

$$(\mathcal{N}_n^{(1)\circ}[\varphi_1], \ldots,\ \mathcal{N}^{(d)\circ}[\varphi_d]), \tag{2.12}$$

converges in distribution to the zero mean Gaussian vector $(G_1, \cdots, G_d) \in \mathbb{R}^d$ with covariance given by

$$\begin{aligned}
&\mathbf{Cov}(G_l, G_p) \\
&= \frac{\sigma^2}{4}(\varphi_l)_1(\varphi_p)_1\Big(\frac{\gamma_{lp}}{\sqrt{\gamma_l\gamma_p}}\Big) + \frac{1}{2}\sum_{k=2}^{\infty} k\,(\varphi_l)_k(\varphi_p)_k\left(\frac{\gamma_{lp}}{\sqrt{\gamma_l\gamma_p}}\right)^k \\
&= \frac{2}{\pi}\oint\limits_{\substack{|z|^2=\gamma_l\\ \mathfrak{Im}\,z>0}}\ \oint\limits_{\substack{|w|^2=\gamma_p\\ \mathfrak{Im}\,w>0}} \varphi_l'\Big(z+\frac{\gamma_l}{z}\Big)\varphi_p'\Big(w+\frac{\gamma_p}{w}\Big)\frac{1}{2\pi}\ln\left|\frac{\gamma_{lp}-zw}{\gamma_{lp}-z\bar{w}}\right|\Big(1-\frac{\gamma_l}{z^2}\Big)\Big(1-\frac{\gamma_p}{w^2}\Big)\,dzdw \\
&\quad + \frac{\gamma_{lp}(\sigma^2-2)}{4\pi^2\gamma_l\gamma_p}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\frac{\lambda\varphi_l(\lambda)}{\sqrt{4\gamma_l-\lambda^2}}d\lambda\int_{-2\sqrt{\gamma_p}}^{2\sqrt{\gamma_p}}\frac{\mu\varphi_p(\mu)}{\sqrt{4\gamma_p-\mu^2}}d\mu.
\end{aligned} \tag{2.13}$$

In the expression for the covariance, $(\varphi_l)_k$ denotes the coefficients in the expansion of φ_l in the (rescaled) Chebyshev basis, i.e.

$$\varphi_l(x) = \sum_{k=0}^{\infty}(\varphi_l)_k T_k^{\gamma_l}(x), \quad (\varphi_l)_k = \frac{2}{\pi}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\varphi_l(t)T_k^{\gamma_l}(t)\frac{dt}{\sqrt{4\gamma_l - t^2}} \tag{2.14}$$

and

$$T_k^{\gamma}(x) = cos\left(k\ arccos\left(\frac{x}{2\sqrt{\gamma}}\right)\right). \tag{2.15}$$

Note the form of the kernel in the above contour integral expression for the covariance. Since it is the Greens function for the Laplacian on $\mathbb{H}$ with Dirichlet boundary conditions (appropriately scaled), we note that the limiting distributions form a family of correlated Gaussian free fields. This is consistent with the previous work of A. Borodin in [6], [7] for the covariance of linear eigenvalue statistics corresponding to polynomial test functions.

Now we formulate our result for the non-Gaussian Wigner matrices.

Theorem 2.2. *Let $W = (W_{jk})_{j,k=1}^{n}$ be an $n \times n$ random matrix and $M = n^{-1/2}W$. Let $\mathcal{B}_1, \ldots \mathcal{B}_d$ be infinite subsets of $\mathbb{N}$ satisfying (2.2-2.4) and (2.5). Assume the following conditions:*
(1) All the entries of W are independent random variables.
(2) The fourth moment of the non-zero off-diagonal entries does not depend on n:

$$\mu_4 = \mathbb{E}\{W_{jk}^4\}$$

(3) There exists a constant σ_6 such that for any j, k, $\mathbb{E}\{|W_{jk}|^6\} < \sigma_6$.
Let $\varphi_1, \cdots, \varphi_d : \mathbb{R} \to \mathbb{R}$ be test functions that satisfy the regularity condition $||\varphi_l||_s < \infty$, for some

$s > 5.5$. *Then the random vector (2.12) converges in distribution to the zero mean Gaussian vector* $(\widetilde{G}_1, \cdots, \widetilde{G}_d) \in \mathbb{R}^d$ *with covariance given by*

$$\mathbf{Cov}(\widetilde{G}_l, \widetilde{G}_p) = \mathbf{Cov}(G_l, G_p) + \frac{\kappa_4 \gamma_{lp}^2}{2\pi^2 \gamma_l^2 \gamma_p^2} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \varphi_l(\lambda) \frac{2\gamma_l - \lambda^2}{\sqrt{4\gamma_l - \lambda^2}} d\lambda \int_{-2\sqrt{\gamma_p}}^{2\sqrt{\gamma_p}} \varphi_p(\mu) \frac{2\gamma_p - \mu^2}{\sqrt{4\gamma_p - \mu^2}} d\mu \tag{2.16}$$

where $\mathbf{Cov}(G_l, G_p)$ *is given by (2.13).*

In the course of the proof of theorem 2.1, it was necessary to understand the following bilinear form.

Definition 2.3. Let M be a Wigner matrix satisfying (1.1), and let $P^{(l)}$, $P^{(l,r)}$ be the projection matrices defined in (2.6) and (2.10). For functions $f, g \in \mathcal{H}_s$, $s > \frac{3}{2}$, define

$$\begin{aligned} \langle f, g \rangle_{lr} \quad &:= \quad \lim_{n\to\infty} \frac{1}{n} \sum_{j,k \in B_l \cap B_r} \mathbb{E}\left[f(M^{(l)})_{jk} \cdot g(M^{(r)})_{kj} \right] \\ &= \quad \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\left[\mathrm{Tr}\left\{ P^{(l)} f(M^{(l)}) \cdot P^{(l,r)} \cdot g(M^{(r)}) P^{(r)} \right\} \right]. \end{aligned} \tag{2.17}$$

Remark 2.4. The bilinear form $\langle \cdot, \cdot \rangle_{lr}$ is well defined on $\mathcal{H}_s \times \mathcal{H}_s$ as a consequence of proposition 3.9. The bilinear form is also well defined for polynomial f and g, see section 3.2 and also lemma 2.5 below.

The following diagonalization lemma is an important technical tool for the proof of theorem 2.1.

Lemma 2.5. *The two families* $\{U_k^{\gamma_l}\}_{k=0}^{\infty}$ *and* $\{U_q^{\gamma_r}\}_{q=0}^{\infty}$ *of rescaled Chebyshev polynomials of the second kind diagonalize the bilinear form (3.131). More precisely,*

$$\frac{1}{\sqrt{\gamma_l \gamma_r}} \, \langle U_k^{\gamma_l}, U_q^{\gamma_r} \rangle_{lr} \ = \ \delta_{kq} \, \left(\frac{\gamma_{lr}}{\sqrt{\gamma_l \gamma_r}} \right)^{k+1}. \tag{2.18}$$

Let $f, g \in \mathcal{H}_s$, *for some* $s > \frac{3}{2}$. *A consequence of (2.18) is that*

$$\langle f, g \rangle_{lr} = \frac{1}{4\pi^2 \gamma_l \gamma_r} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} f(x) g(y) \left[\sum_{k=0}^{\infty} U_k^{\gamma_l}(x) U_k^{\gamma_r}(y) \frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2} \gamma_r^{k/2}} \right] \sqrt{4\gamma_l - x^2} \sqrt{4\gamma_r - y^2} dy dx. \tag{2.19}$$

In section 3.2 it will also be proved that, with f, g given as above, almost surely

$$\begin{aligned} &\lim_{n\to\infty} \frac{1}{n} \mathrm{Tr}\left\{ P^{(l)} f(M^{(l)}) \cdot P^{(l,r)} \cdot g(M^{(r)}) P^{(r)} \right\} \\ &= \quad \frac{1}{4\pi^2 \gamma_l \gamma_r} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} f(x) g(y) \left[\sum_{k=0}^{\infty} U_k^{\gamma_l}(x) U_k^{\gamma_r}(y) \frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2} \gamma_r^{k/2}} \right] \sqrt{4\gamma_l - x^2} \sqrt{4\gamma_r - y^2} dy dx. \end{aligned} \tag{2.20}$$

Remark 2.6. Recall that the rescaled Chebyshev polynomials of the second kind are orthonormal with respect to the Wigner semicircle law, i.e.

$$\frac{1}{2\pi\gamma}\int_{-2\sqrt{\gamma}}^{2\sqrt{\gamma}} U_k^{\gamma}(x)U_q^{\gamma}(x)\sqrt{4\gamma - x^2}dx = \delta_{kq}. \tag{2.21}$$

Also,

$$U_k^{\gamma}(2\sqrt{\gamma}\cos(\theta)) = \frac{\sin((k+1)\theta)}{\sin(\theta)}. \tag{2.22}$$

The paper is structured as follows. The proof of theorem 2.1 appears in section 3 and the proof of theorem 2.2 appears in section 4.

3 Proof of Theorem 2.1

3.1 Stein-Tikhomirov Method

We follow the approach used by A. Lytova and L. Pastur in [10] for the full Wigner matrix case. Essentially, it is a modification of the Stein-Tikhomirov method. This approach was also used to prove the CLT for linear eigenvalue statistics of band random matrices in [9], which is connected to our work through the Chu-Vandermonde identity. See section 3.2. While several steps of our proof are similar to the ones in [10], the fact that we are dealing with submatrices introduces new technical difficulties.

We will prove Theorem 2.1 in the present section and extend to non-Gaussian Wigner matrices later. The following inequalities will be used often. A consequence of the Poincaré inequality is that, for differentiable test functions φ,

$$\begin{aligned}\mathbf{Var}\{\mathrm{Tr}\varphi(M)\} &\leq \frac{4(\sigma^2+1)}{n}\mathbb{E}\left[\mathrm{Tr}\{\varphi'(M)(\varphi'(M))^*\}\right] &(3.1)\\ &\leq 4(\sigma^2+1)\left(\sup_{x\in\mathbb{R}}|\varphi'(x)|\right)^2. &(3.2)\end{aligned}$$

See [10] for a reference. The next inequality is due to M. Shcherbina, see [11]. Let $s > 3/2$ and $\varphi \in \mathcal{H}_s$. Then there is a constant $C_s > 0$, so that

$$\mathbf{Var}\{\mathrm{Tr}\varphi(M)\} \leq C_s||\varphi||_s^2. \tag{3.3}$$

Let $\epsilon > 0$ and set $s = \frac{5}{2} + \epsilon$. Recall that the regularity assumption on the test functions is that $||\varphi_l||_{5/2+\epsilon} < \infty$, for $1 \leq l \leq d$. There exists a $C_\epsilon > 0$ so that

$$\mathbf{Var}\{\mathcal{N}^{(l)}[\varphi_l]\} = \mathbf{Var}\{\mathrm{Tr}\varphi_l(M(B_l))\} \leq C_\epsilon||\varphi_l||_{5/2+\epsilon}^2. \tag{3.4}$$

The inequality holds because of (3.3), noting that $M(B_l)$ is an ordinary $|B_l| \times |B_l|$ Gaussian Wigner matrix. Note that this bound is n-independent.

It is sufficient to prove the CLT for all linear combinations of the components of (2.12). Consider a linear combination $\xi := \sum_{l=1}^d \alpha_l\mathcal{N}^{(l)\circ}[\varphi_l]$, and denote the characteristic function by

$$Z_n(x) = \mathbb{E}[e^{ix\xi}]. \tag{3.5}$$

It is a basic fact that the characteristic function of the Gaussian distribution with variance V is given by

$$Z(x) := e^{-x^2V/2}. \tag{3.6}$$

As a consequence of the Levy Continuity theorem, to prove theorem 2.1 it will be sufficient to demonstrate that for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} Z_n(x) = Z(x), \tag{3.7}$$

where $Z(x)$ is given as above with

$$V := \lim_{n\to\infty} \left[\sum_{l=1}^{d} \alpha_l^2 \mathbf{Var}\left(\mathcal{N}_n^{(l)\circ}[\varphi_l]\right) + 2 \sum_{1\le l<r\le d} \alpha_l\alpha_r \mathbf{Cov}\left(\mathcal{N}_n^{(l)\circ}[\varphi_l],\ \mathcal{N}_n^{(r)\circ}[\varphi_r]\right)\right]. \tag{3.8}$$

So V is the limiting variance of ξ. It will be demonstrated that $Z_n(x)$ converges uniformly to the solution of the following equation

$$Z(x) \ = \ 1 - V \int_0^x yZ(y)dy. \tag{3.9}$$

Note that (3.6) is the unique solution of (3.9) within the class of bounded and continuous functions. Therefore, to prove the theorem, it is sufficient to demonstrate that the pointwise limit of $Z_n(x)$ is a continuous and bounded function which satisfies equation (3.9), with V given by (3.8).

Observe that

$$Z_n'(x) = i\mathbb{E}[\xi e^{ix\xi}] = i\sum_{l=1}^{d} \alpha_l \mathbb{E}\{\mathcal{N}_n^{(l)\circ}[\varphi_l]e^{ix\xi}\}. \tag{3.10}$$

Now it follows by the Cauchy-Schwartz inequality and (3.4) that

$$\left|Z_n'(x)\right| \ \le \ \sum_{l=1}^{d} |\alpha_l| \sqrt{\mathbf{Var}\{\mathcal{N}^{(l)}[\varphi_l]\}} \le Const \sum_{l=1}^{d} |\alpha_l|\ ||\varphi_l||_{5/2+\epsilon}. \tag{3.11}$$

Since $Z_n(0) = 1$, we have by the fundamental theorem of calculus that

$$Z_n(x) = 1 + \int_0^x Z_n'(y)dy. \tag{3.12}$$

Then to prove the CLT it is sufficient to show that any uniformly converging subsequences $\{Z_{n_m}\}$ and $\{Z_{n_m}'\}$, satisfy

$$\lim_{n_m\to\infty} Z_{n_m}(x) = Z(x), \tag{3.13}$$

and

$$\lim_{n_m\to\infty} Z_{n_m}'(x) = -xVZ(x). \tag{3.14}$$

A pre-compactness argument based on the Arzela-Ascoli theorem will be developed below, which ensures that the subsequences converge uniformly, implying that the limit is a continuous function.

The estimate $|Z_n(x)| \leq 1$, for all n, shows that the sequence is uniformly bounded. Generally we will abuse the subsequence notation by writing $\{n\}$ for a uniformly converging subsequence. Since (3.11) combined with $||\varphi_l||_{5/2+\epsilon} < \infty$ justify an application of the dominated convergence theorem in (3.12), it follows from (3.13) and (3.14) that the limit of $Z_n(x)$ satisfies equation (3.9). Therefore the pointwise limit (3.7) holds. We turn our attention to the pre-compactness argument, and will argue later that (3.13) and (3.14) hold. Similar notation is used as in [10]. Denote by

$$D_{jk} := \partial/\partial M_{jk}; \tag{3.15}$$

$$U^{(l)}(t) := e^{itM^{(l)}},\ U^{(l)}_{jk}(t) := (U^{(l)}(t))_{jk}; \tag{3.16}$$

$$u^{(l)}_n(t) := \mathrm{Tr}\{P^{(l)}U^{(l)}(t)P^{(l)}\},\ u^{(l)\circ}_n(t) := u^{(l)}_n(t) - \mathbb{E}\{u^{(l)}_n(t)\}. \tag{3.17}$$

For the benefit of the reader, what is needed is recorded below. Recall that $U^{(l)}(t)$ is a unitary matrix, and writing $\beta_{jk} := (1+\delta_{jk})^{-1}$, we have

$$|U^{(l)}_{jk}| \leq 1,\ \sum_{k=1}^{n} |U^{(l)}_{jk}|^2 = 1,\ \|U^{(l)}\| = 1. \tag{3.18}$$

Moreover,

$$D_{jk}U^{(l)}_{ab}(t) = i\beta_{jk}\mathbf{1}_{\{j,k\in B_l\}}\left(U^{(l)}_{aj} * U^{(l)}_{bk}(t) + U^{(l)}_{ak} * U^{(l)}_{bj}(t)\right), \tag{3.19}$$

where

$$f * g(t) := \int_0^t f(y)g(t-y)dy. \tag{3.20}$$

Applying the Fourier inversion formula

$$\varphi_l(\lambda) = \int_{-\infty}^{\infty} e^{it\lambda}\widehat{\varphi}_l(t)dt, \tag{3.21}$$

it follows that

$$\mathcal{N}^{(l)\circ}[\varphi_l] = \int_{-\infty}^{\infty} \widehat{\varphi}_l(t)u^{(l)\circ}_n(t)dt. \tag{3.22}$$

Now define

$$e_n(x) := e^{ix\xi}. \tag{3.23}$$

Using the Fourier representation of the linear eigenvalue statistics in (3.10), it follows that

$$Z'_n(x) = i\sum_{l=1}^{d}\alpha_l\int_{-\infty}^{\infty}\widehat{\varphi}_l(t)Y^{(l)}_n(x,t)dt, \tag{3.24}$$

where

$$Y^{(l)}_n(x,t) := \mathbb{E}\left[u^{(l)\circ}_n(t)e_n(x)\right]. \tag{3.25}$$

The limit of $Y^{(l)}_n(x,t)$ is determined later in the proof. Since

$$\overline{Y^{(l)}_n(x,t)} = Y^{(l)}_n(-x,-t), \tag{3.26}$$

we need only consider $t \geq 0$. It will now be demonstrated that each sequence $\{Y_n^{(l)}\}$ is bounded and equicontinuous on compact subsets of $\{x \in \mathbb{R}, t \geq 0\}$, and that every uniformly converging subsequence has the same limit $Y^{(l)}$, implying (3.13) and (3.14). See proposition 3.1.

Let $\varphi(x) = e^{itx}$, and note that $\sup_{x\in\mathbb{R}} |\varphi'(x)| = |t|$. Applying the inequality (3.2) to the linear eigenvalue statistic $\mathcal{N}^{(l)}[\varphi]$, we obtain

$$\mathbf{Var}\{u_n^{(l)}(t)\} = \mathbf{Var}\{\mathcal{N}^{(l)}[\varphi]\} \leq 4(\sigma^2+1)t^2. \tag{3.27}$$

Now set $\varphi(x) = ixe^{itx}$, and notice that

$$\frac{d}{dt}u_n^{(l)}(t) = i\mathrm{Tr}\{M^{(l)}e^{itM^{(l)}}\}.$$

Using the inequality (3.1) and the fact that $n^{-1}\mathbb{E}\mathrm{Tr}(M^{(l)})^2 \leq \sigma^2+1$, it follows that

$$\begin{aligned}
\mathbf{Var}\{\frac{d}{dt}u_n^{(l)}(t)\} &\leq \frac{4(\sigma^2+1)}{n}\mathbb{E}\left[\mathrm{Tr}\{\varphi'(M^{(l)})(\varphi'(M^{(l)}))^*\}\right] \\
&\leq \frac{4(\sigma^2+1)}{n}\mathbb{E}\left[\mathrm{Tr}\{1+t^2(M^{(l)})^2\}\right] \\
&\leq 4(\sigma^2+1)[1+(\sigma^2+1)t^2].
\end{aligned} \tag{3.28}$$

Using the Cauchy-Schwartz inequality, the bound $|e_n(x)| \leq 1$, (3.27) and (3.28), we obtain

$$\left|Y_n^{(l)}(x,t)\right| \leq \mathbf{Var}^{1/2}\{u_n^{(l)}(t)\} \leq 2(\sigma^2+1)^{1/2}|t|, \tag{3.29}$$

and also

$$\left|\frac{\partial}{\partial t}Y_n^{(l)}(x,t)\right| \leq \mathbf{Var}^{1/2}\{\frac{d}{dt}u_n^{(l)}(t)\} \leq 2\sqrt{(\sigma^2+1+(\sigma^2+1)^2t^2)}. \tag{3.30}$$

Observe that

$$\frac{d}{dx}e_n(x) = ie_n(x)\sum_{r=1}^{d}\alpha_r\,\mathcal{N}^{(r)\circ}[\varphi_r].$$

Using the above derivative with the Cauchy-Schwartz inequality, (3.4) and (3.27), we have that

$$\begin{aligned}
\left|\frac{\partial}{\partial x}Y_n^{(l)}(x,t)\right| &= \left|i\sum_{r=1}^{d}\alpha_r\;\mathbb{E}[u_n^{(l)\circ}(t)\mathcal{N}_n^{(r)\circ}[\varphi_r]e_n(x)]\right| \\
&\leq \mathbf{Var}^{1/2}\{u_n^{(l)}(t)\}\sum_{r=1}^{d}|\alpha_r|\;\mathbf{Var}^{1/2}\{\mathcal{N}^{(r)}[\varphi_r]\} \\
&\leq Const\cdot|t|\sum_{r=1}^{d}|\alpha_r|\;||\varphi_r||_{5/2+\epsilon}.
\end{aligned} \tag{3.31}$$

It follows from (3.29), the mean value theorem combined with (3.30) and (3.31), and $||\varphi_r||_{5/2+\epsilon} < \infty$, that each sequence $Y_n^{(l)}(x,t)$ is bounded and equicontinuous on compact subsets of $\mathbb{R}^2$. The following proposition justifies this restriction.

Proposition 3.1. *In order to prove the functions $Y_n^{(l)}(x,t)$ converge uniformly to appropriate limits so that (3.24) implies (3.14), it is sufficient to prove the convergence of $Y_n^{(l)}(x,t)$ on arbitrary compact subsets of $\{x \in \mathbb{R}, t \geq 0\}$.*

Proof. Let $\delta > 0$. Recall that the regularity assumption on the test functions φ_l are

$$\int_{\mathbb{R}} (1+|h|)^{5+\epsilon} |\widehat{\varphi}_l(h)|^2 dh < \infty,$$

i.e. that $\varphi_l \in \mathcal{H}_s$, with $s = 5/2 + \epsilon$. Using the Cauchy-Schwartz inequality, it follows that

$$\int_{\mathbb{R}} (1+|h|)|\widehat{\varphi}_l(h)|dh \leq \sqrt{\int_{\mathbb{R}} \frac{dh}{(1+|h|)^{3+\epsilon}}} \cdot \sqrt{\int_{\mathbb{R}} (1+|h|)^{5+\epsilon}|\widehat{\varphi}_l(h)|^2 dh}, \tag{3.32}$$

which implies that

$$\int_{\mathbb{R}} |h| \cdot |\widehat{\varphi}_l(h)|dh < \infty. \tag{3.33}$$

A consequence of the finiteness of the integral in (3.33), for each $1 \leq l \leq d$, is that there exists a $T > 0$ so that

$$2(\sigma^2+1)^{1/2} \sum_{l=1}^{d} |\alpha_l| \int_{|t|\geq T} |t| \cdot |\widehat{\varphi}_l(t)|dt < \delta. \tag{3.34}$$

Using (3.24), we can write

$$Z_n'(x) = i \sum_{l=1}^{d} \alpha_l \int_{-T}^{T} \widehat{\varphi}_l(t) Y_n^{(l)}(x,t)dt + i \sum_{l=1}^{d} \alpha_l \int_{|t|\geq T} \widehat{\varphi}_l(t) Y_n^{(l)}(x,t)dt. \tag{3.35}$$

Then (3.35), (3.29), (3.34) imply that

$$\begin{aligned} \left| Z_n'(x) - i \sum_{l=1}^{d} \alpha_l \int_{-T}^{T} \widehat{\varphi}_l(t) Y_n^{(l)}(x,t)dt \right| &\leq \sum_{l=1}^{d} |\alpha_l| \int_{|t|\geq T} |\widehat{\varphi}_l(t)| \cdot |Y_n^{(l)}(x,t)|dt \\ &\leq 2(\sigma^2+1)^{1/2} \sum_{l=1}^{d} |\alpha_l| \int_{|t|\geq T} |t| \cdot |\widehat{\varphi}_l(t)|dt \\ &< \delta. \end{aligned} \tag{3.36}$$

Notice that the estimate (3.36) is n-independent, so that in particular the estimate holds in the limit $n \to \infty$. Since δ was arbitrary, this completes the proof of the proposition. □

This completes the pre-compactness argument, which allows us to pass to the limit in (3.24) and in (3.12), and conclude that $Z_n(x)$ converges pointwise to the unique solution of equation (3.9) belonging to $C_b(\mathbb{R})$, implying (3.7), and hence the conclusion of the theorem. Now we show the limiting behavior of the sequences $Y_n^{(l)}(x,t)$ imply (3.13) and (3.14). Consider the identity

$$e^{itM^{(l)}} = I + i\int_0^t M^{(l)} e^{ihM^{(l)}} dh.$$

Apply this identity, noting that $M^{(l)}_{jk} = 0,$ if $j, k \notin B_l$, to obtain that

$$\begin{aligned} u^{(l)\circ}_n(t) &= \mathrm{Tr}\{P^{(l)}U^{(l)}(t)P^{(l)}\} - \mathbb{E}[\mathrm{Tr}\{P^{(l)}U^{(l)}(t)P^{(l)}\}] \\ &= i\int_0^t \sum_{j,k=1}^n \left[M^{(l)}_{jk}U^{(l)}_{jk}(t_1) - \mathbb{E}[M^{(l)}_{jk}U^{(l)}_{jk}(t_1)]\right]. \end{aligned} \tag{3.37}$$

Recalling that $Y^{(l)}_n(x,t) = \mathbb{E}\left[u^{(l)\circ}_n(t)e_n(x)\right]$, and applying the decoupling formula for Gaussian random variables, it follows from (3.37) that

$$\begin{aligned} Y^{(l)}_n(x,t) &= i\int_0^t \sum_{j,k=1}^n \mathbb{E}[M^{(l)}_{jk}U^{(l)}_{jk}(t_1)e^\circ_n(x)]dt_1 \\ &= \frac{2i}{n}\int_0^t \sum_{1\le j<k\le n} \mathbf{1}_{\{j,k\in B_l\}}\ \mathbb{E}\left[D_{jk}U^{(l)}_{jk}(t_1)e^\circ_n(x)\right]dt_1. \\ &\quad + \frac{i\sigma^2}{n}\int_0^t \sum_{j=1}^n \mathbf{1}_{\{j\in B_l\}}\mathbb{E}\left[D_{jj}U^{(l)}_{jj}(t_1)e^\circ_n(x)\right]dt_1. \end{aligned} \tag{3.38}$$

It will be useful to rewrite (3.38) as

$$\begin{aligned} Y^{(l)}_n(x,t) &= \underbrace{\frac{i}{n}\int_0^t \sum_{j,k=1}^n \mathbf{1}_{\{j,k\in B_l\}}(1+\delta_{jk})\mathbb{E}\left[D_{jk}U^{(l)}_{jk}(t_1)e^\circ_n(x)\right]dt_1}_{=:T_1} \\ &\quad \underbrace{+\frac{i(\sigma^2-2)}{n}\int_0^t \sum_{j=1}^n \mathbf{1}_{\{j\in B_l\}}\mathbb{E}\left[D_{jj}U^{(l)}_{jj}(t_1)e^\circ_n(x)\right]dt_1\,.}_{=:T_2} \end{aligned} \tag{3.39}$$

The reason for the rewrite is that it splits the functions $Y^{(l)}_n(x,t)$ into a part that depends on the distribution of the diagonal entries and a part that corresponds to the same term as for the Gaussian Orthogonal Ensemble, for which $\sigma^2 = 2$. Recalling that $e_n(x)$ is given by (3.23), again writing $\beta_{jk} = (1+\delta_{jk})^{-1}$ and using the identity

$$D_{jk}\mathrm{Tr}f(M) = 2\beta_{jk}f'(M)_{jk},$$

it follows by a direct calculation that

$$D_{jk}e_n(x) = 2i\beta_{jk}xe_n(x)\sum_{r=1}^d \alpha_r\left(P^{(r)}\varphi'_r(M^{(r)})P^{(r)}\right)_{jk}. \tag{3.40}$$

Then for $1 \le l \le d$, using (3.40) and (3.19), it follows that

$$\begin{aligned} T_1 &= \frac{-1}{n}\int_0^t\int_0^{t_1}\mathbb{E}\left[\sum_{j,k=1}^n \mathbf{1}_{\{j,k\in B_l\}}U_{jj}^{(l)}(t_2)U_{kk}^{(l)}(t_1-t_2)e_n^\circ(x)\right]dt_2dt_1 \\ &\quad -\frac{1}{n}\int_0^t\int_0^{t_1}\mathbb{E}\left[\sum_{j,k=1}^n \mathbf{1}_{\{j,k\in B_l\}}U_{jk}^{(l)}(t_2)U_{jk}^{(l)}(t_1-t_2)e_n^\circ(x)\right]dt_2dt_1 \\ &\quad -\frac{2x}{n}\int_0^t\mathbb{E}\left[\sum_{j,k=1}^n \mathbf{1}_{\{j,k\in B_l\}}U_{jk}^{(l)}(t_1)e_n(x)\sum_{r=1}^d \alpha_r\left(P^{(r)}\varphi_r'(M^{(r)})P^{(r)}\right)_{jk}\right]dt_1, \end{aligned} \tag{3.41}$$

and also that

$$\begin{aligned} T_2 &= \underbrace{\frac{-(\sigma^2-2)}{n}\int_0^t\int_0^{t_1}\mathbb{E}\left[\sum_{j=1}^n \mathbf{1}_{\{j\in B_l\}}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)e_n^\circ(x)\right]dt_2dt_1}_{=:T_{21}} \\ &\quad \underbrace{-\frac{(\sigma^2-2)x}{n}\int_0^t\mathbb{E}\left[\sum_{j=1}^n \mathbf{1}_{\{j\in B_l\}}U_{jj}^{(l)}(t_1)e_n(x)\sum_{r=1}^d \alpha_r\left(P^{(r)}\varphi_r'(M^{(r)})P^{(r)}\right)_{jj}\right]dt_1}_{=:T_{22}}. \end{aligned} \tag{3.42}$$

Using the semigroup property

$$U^{(l)}(t)U^{(l)}(h) = U^{(l)}(t+h),$$

it follows form (3.41) that T_1 can be written

$$\begin{aligned} T_1 &= \underbrace{-\frac{1}{n}\int_0^t\int_0^{t_1}\mathbb{E}\left[u_n^{(l)}(t_1-t_2)u_n^{(l)}(t_2)e_n^\circ(x)\right]dt_2dt_1}_{=:T_{11}} \\ &\quad \underbrace{-\frac{1}{n}\int_0^t t_1\mathbb{E}\left[u_n^{(l)}(t_1)e_n^\circ(x)\right]dt_1}_{=:T_{12}} \\ &\quad \underbrace{-\frac{2x}{n}\sum_{r=1}^d\alpha_r\int_0^t\mathbb{E}\left[\mathrm{Tr}\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\}e_n(x)\right]dt_1}_{=:T_{13}}. \end{aligned} \tag{3.43}$$

Define

$$\bar{v}_n^{(l)}(t) := \frac{1}{n}\mathbb{E}[u_n^{(l)}(t)]. \tag{3.44}$$

The following proposition presents the functions $Y_n^{(l)}(x,t)$ in a form that is amenable to asymptotic analysis.

Proposition 3.2. *The equation $Y_n^{(l)}(x,t) = T_1 + T_2$, can be written as*

$$Y_n^{(l)}(x,t) + 2\int_0^t \int_0^{t_1} \bar{v}_n^{(l)}(t_1 - t_2) Y_n^{(l)}(x,t_2) dt_2 dt_1 = xZ_n(x)\left[A_n^{(l)}(t) + Q_n^{(l)}(t)\right] + r_n^{(l)}(x,t), \tag{3.45}$$

where

$$A_n^{(l)}(t) := -2\sum_{r=1}^{d} \alpha_r \int_0^t \frac{1}{n}\mathbb{E}\left[Tr\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\}\right] dt_1, \tag{3.46}$$

$$Q_n^{(l)}(t) := \frac{-(\sigma^2-2)}{n}\sum_{r=1}^{d} \alpha_r \int_0^t \sum_{j=1}^{n} \mathbf{1}_{\{j\in B_l\cap B_r\}}\mathbb{E}\left[U_{jj}^{(l)}(t_1)\varphi_r'(M^{(r)})_{jj}\}\right] dt_1, \tag{3.47}$$

and

$$
\begin{aligned}
r_n^{(l)}(x,t) \quad = \quad & \\
& \frac{-1}{n}\int_0^t t_1 Y_n^{(l)}(x,t_1) dt_1 && (3.48)\\
& -\frac{1}{n}\int_0^t \int_0^{t_1} \mathbb{E}\left[u_n^{(l)\circ}(t_1-t_2)u_n^{(l)\circ}(t_2)e_n^\circ(x)\right] dt_2 dt_1 && (3.49)\\
& -\frac{2x}{n}\sum_{r=1}^{d}\alpha_r \int_0^t \mathbb{E}\left[Tr\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\}e_n^\circ(x)\right] dt_1 && (3.50)\\
& -\frac{(\sigma^2-2)}{n}\int_0^t \int_0^{t_1} \mathbb{E}\left[\sum_{j=1}^{n}\mathbf{1}_{\{j\in B_l\}}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)e_n^\circ(x)\right] dt_2 dt_1 && (3.51)\\
& -\frac{x(\sigma^2-2)}{n}\sum_{r=1}^{d}\alpha_r \int_0^t \sum_{j=1}^{n}\mathbf{1}_{\{j\in B_l\cap B_r\}}\mathbb{E}\left[U_{jj}^{(l)}(t_1)\varphi_r'(M^{(r)})_{jj}e_n^\circ(x)\right] dt_1. && (3.52)
\end{aligned}
\tag{3.53}$$

Proof. Begin with the term T_{11}, defined in (3.43). Write

$$T_{11} = -\frac{1}{n}\int_0^t \int_0^{t_1} \mathbb{E}\left[\left(u_n^{(l)\circ}(t_1-t_2) + n\bar{v}_n(t_1-t_2)\right)\cdot\left(u_n^{(l)\circ}(t_2) + n\bar{v}_n(t_2)\right)e_n^\circ(x)\right] dt_2 dt_1, \tag{3.54}$$

so that

$$
\begin{aligned}
T_{11} \quad = \quad & \\
& -\frac{1}{n}\int_0^t \int_0^{t_1} \mathbb{E}\left[u_n^{(l)\circ}(t_1-t_2)u_n^{(l)\circ}(t_2)e_n^\circ(x)\right] dt_2 dt_1\\
& -\int_0^t \int_0^{t_1} \bar{v}_n(t_1-t_2)\mathbb{E}\left[u_n^{(l)\circ}(t_2)e_n^\circ(x)\right] dt_2 dt_1\\
& -\int_0^t \int_0^{t_1} \bar{v}_n(t_2)\mathbb{E}\left[u_n^{(l)\circ}(t_1-t_2)e_n^\circ(x)\right] dt_2 dt_1\\
& -n\int_0^t \int_0^{t_1} \bar{v}_n(t_1-t_2)\cdot\bar{v}_n(t_2)\underbrace{\mathbb{E}\left[e_n^\circ(x)\right]}_{=0} dt_2 dt_1.
\end{aligned}
\tag{3.55}$$

Noting that

$$\mathbb{E}\left[u_n^{(l)\circ}(t_2)e_n^\circ(x)\right] = Y_n^{(l)}(x,t_2),\ \mathbb{E}\left[u_n^{(l)\circ}(t_1-t_2)e_n^\circ(x)\right] = Y_n^{(l)}(x,t_1-t_2),$$

and also that

$$\int_0^t\int_0^{t_1} \bar{v}_n(t_2)Y_n^{(l)}(x,t_1-t_2)dt_2dt_1 = \int_0^t\int_0^{t_1} \bar{v}_n(t_1-t_2)Y_n^{(l)}(x,t_2)dt_2dt_1,$$

it follows that

$$\begin{aligned} T_{11} &= \\ &-\frac{1}{n}\int_0^t\int_0^{t_1}\mathbb{E}\left[u_n^{(l)\circ}(t_1-t_2)u_n^{(l)\circ}(t_2)e_n^\circ(x)\right]dt_2dt_1 &(3.56)\\ &-2\int_0^t\int_0^{t_1}\bar{v}_n(t_1-t_2)Y_n^{(l)}(x,t_2)dt_2dt_1. &(3.57)\\ & &(3.58)\end{aligned}$$

The term (3.56) goes into the remainder, which becomes (3.49). Also, (3.57) is added to the left-hand side of (3.45). Now consider the term T_{12}, defined in (3.43). We have that

$$T_{12} = -\frac{1}{n}\int_0^t t_1Y_n^{(l)}(x,t_1)dt_1, \tag{3.59}$$

which becomes (3.48) in the remainder. Consider the term T_{13}, also defined in (3.43). Writing

$$T_{13} = -\frac{2x}{n}\sum_{r=1}^d \alpha_r\int_0^t \mathbb{E}\left[\mathrm{Tr}\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\}\cdot(e_n^\circ(x)+Z_n(x))\right]dt_1, \tag{3.60}$$

it follows, with $A_n^{(l)}(t)$ given by (3.46), that

$$\begin{aligned} T_{13} &= \\ &-\frac{2x}{n}\sum_{r=1}^d \alpha_r\int_0^t \mathbb{E}\left[\mathrm{Tr}\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\}e_n^\circ(x)\right]dt_1 &(3.61)\\ &+xZ_n(x)A_n^{(l)}(t). &(3.62)\\ & &(3.63)\end{aligned}$$

Then (3.61) becomes (3.50) in the remainder, while (3.62) remains on the right-hand side of (3.45).

Now consider the term T_{21}, defined in (3.42). This term becomes (3.51) in the remainder. Finally, consider the term T_{22}, also defined in (3.42). Write

$$T_{22} = -\frac{(\sigma^2-2)x}{n}\int_0^t \mathbb{E}\left[\sum_{j=1}^n \mathbf{1}_{\{j\in B_l\}}U_{jj}^{(l)}(t_1)\cdot(e_n^\circ(x)+Z_n(x))\sum_{r=1}^d\alpha_r\left(P^{(r)}\varphi_r'(M^{(r)})P^{(r)}\right)_{jj}\right]dt_1, \tag{3.64}$$

so that, with $Q_n^{(l)}(t)$ given by (3.47) ,

$$\begin{aligned} T_{22} &= \\ & -\frac{(\sigma^2-2)x}{n}\int_0^t \mathbb{E}\left[\sum_{j=1}^n \mathbf{1}_{\{j\in B_l\}} U_{jj}^{(l)}(t_1) e_n^\circ(x) \sum_{r=1}^d \alpha_r \left(P^{(r)}\varphi_r'(M^{(r)})P^{(r)}\right)_{jj}\right] dt_1 \quad (3.65) \\ & + xZ_n(x)\cdot Q_n^{(l)}(t). \quad (3.66) \\ & \quad (3.67) \end{aligned}$$

The term (3.65) becomes (3.52) in the remainder. Also, the term (3.66) remains on the right-hand side of (3.45). This completes the argument for proposition 3.2. □

We now turn our attention to the remainder term, $r_n^{(l)}(x,t)$, of proposition 3.2. The content of the following proposition is that the remainder is negligible in the limit.

Proposition 3.3. *Each term of $r_n^{(l)}(x,t)$ converges to 0 uniformly on compact subsets of $\{x\in\mathbb{R}, t\geq 0\}$, for $1\leq l\leq d$. In other words, we have the uniform limit*

$$\lim_{n\to\infty} r_n^{(l)}(x,t) = 0. \quad (3.68)$$

Proof. Begin with the term (3.48). Applying the estimate (3.29), we obtain

$$\begin{aligned} \left|\frac{1}{n}\int_0^t t_1 Y_n^{(l)}(x,t_1)dt_1\right| &\leq \frac{1}{n}t^2\left|Y_n^{(l)}(x,t)\right| \\ &\leq \frac{2(\sigma^2+1)^{1/2}}{n}|t|^3 \\ &= O\left(\frac{1}{n}\right). \end{aligned} \quad (3.69)$$

Now consider the term (3.49). Using the bound $|e_n^\circ(x)|\leq 2$, the Cauchy-Schwartz inequality, and (3.27) twice, it follows that

$$\begin{aligned} \left|\frac{1}{n}\int_0^t\int_0^{t_1}\mathbb{E}\left[u_n^{(l)\circ}(t_1-t_2)u_n^{(l)\circ}(t_2)e_n^\circ(x)\right]dt_2dt_1\right| &\leq \frac{2}{n}t^2\mathbf{Var}^{1/2}\{u_n^{(l)}(t)\}\mathbf{Var}^{1/2}\{u_n^{(l)}(t)\} \\ &\leq \frac{8(\sigma^2+1)^{1/2}}{n}t^4 \\ &= O\left(\frac{1}{n}\right). \end{aligned} \quad (3.70)$$

Consider the term (3.50) next. Applying (2.19) of lemma 2.5 to the exponential function and φ_r', and noting that $\varphi_r'\in\mathcal{H}_{\frac{3}{2}+\epsilon}$, it follows that

$$\begin{aligned} &\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\mathrm{Tr}\left\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\right\}\right] \\ &= \frac{1}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} e^{it_1x}\varphi_r'(y)\left[\sum_{k=0}^\infty U_k^{\gamma_l}(x)U_k^{\gamma_r}(y)\frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2}\gamma_r^{k/2}}\right]\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}dydx. \end{aligned} \quad (3.71)$$

While the exponential function does not belong to $\mathcal{H}_{\frac{3}{2}+\epsilon}$, we can truncate the exponential function in a smooth fashion outside the support of the semicircle law, so that the truncated exponential function belongs to $\mathcal{H}_{\frac{3}{2}+\epsilon}$. We may replace the exponential function by its truncated version because the eigenvalues of the submatrices concentrate in the support of the semicircle law with overwhelming probability. Then

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\right\}$$
$$=\frac{1}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}e^{it_1x}\varphi_r'(y)\left[\sum_{k=0}^{\infty}U_k^{\gamma_l}(x)U_k^{\gamma_r}(y)\frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2}\gamma_r^{k/2}}\right]\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}dydx. \tag{3.72}$$

Here it is not so important to know the exact value of the limit, but we will use the fact that we have convergence in the mean and almost surely to the same limit. Note the convergence in (3.71) implies that the sequence of numbers

$$\frac{1}{n}\mathbb{E}\left[\mathrm{Tr}\left\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\right\}\right],$$

is bounded. Also the convergence in (3.72) implies that the random variables

$$\frac{1}{n}\mathrm{Tr}\left\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\right\},$$

are bounded with probability 1. Using (3.71) and (3.72) with the dominated convergence theorem, it now follows that

$$\lim_{n\to\infty}\mathbb{E}\left|\frac{1}{n}\mathrm{Tr}P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}-\frac{1}{n}\mathbb{E}\left\{\mathrm{Tr}P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\right\}\right|=0. \tag{3.73}$$

Combining the bound $|e_n(x)|\le 1$ with (3.73), it follows that

$$\begin{aligned}
&\left|\frac{1}{n}\mathbb{E}\left[\mathrm{Tr}\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\}e_n^\circ(x)\right]\right|\\
=&\left|\mathbb{E}\left[\left(\frac{1}{n}\mathrm{Tr}P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}-\frac{1}{n}\mathbb{E}\left\{\mathrm{Tr}P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\right\}\right)e_n(x)\right]\right|\\
\le&\ \mathbb{E}\left|\frac{1}{n}\mathrm{Tr}P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}-\frac{1}{n}\mathbb{E}\left\{\mathrm{Tr}P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\right\}\right|\to 0.
\end{aligned} \tag{3.74}$$

Then, using (3.74) in the remainder term (3.50), it follows that

$$\left|-\frac{2x}{n}\sum_{r=1}^{d}\alpha_r\int_0^t\mathbb{E}\left[\mathrm{Tr}\{P^{(l)}U^{(l)}(t_1)P^{(l,r)}\varphi_r'(M^{(r)})P^{(r)}\}e_n^\circ(x)\right]dt_1\right|\to 0 \text{ as } n\to\infty. \tag{3.75}$$

Consider (3.51), which is the next term in the remainder. Observe that, again using the Cauchy-Schwartz inequality and the fact that $|e_n(x)| \le 1$,

$$\begin{aligned}
&\mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{\{j\in B_l\}}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)e_n^{\circ}(x)\right]\\
&= \mathbb{E}\left[\frac{1}{n}\sum_{j\in B_l}^{n}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)e_n^{\circ}(x)\right]\\
&\le \mathbb{E}\left|\frac{1}{n}\sum_{j\in B_l}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)-\frac{1}{n}\mathbb{E}\left\{\sum_{j\in B_l}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)\right\}\right|\\
&\le \mathbf{Var}^{1/2}\left\{\frac{1}{n}\sum_{j\in B_l}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)\right\}.
\end{aligned}\tag{3.76}$$

For fixed $j,p,q\in B_l$, using (3.19),

$$\begin{aligned}
D_{pq}U_{jj}^{(l)}(t) &= i\beta_{pq}\left[U_{jp}^{(l)}*U_{jq}^{(l)}(t)+U_{jp}^{(l)}*U_{jq}^{(l)}(t)\right]\\
&= 2i\beta_{pq}\int_0^t U_{jp}^{(l)}(t-h)U_{jq}^{(l)}(h)dh.
\end{aligned}\tag{3.77}$$

Using (3.77), recalling that $\beta_{pq}=(1+\delta_{pq})^{-1}\le 1$, and the Cauchy-Schwartz inequality, it follows that

$$\left|D_{pq}U_{jj}^{(l)}(t)\right|^2 \le 4|t|\int_0^t |U_{jp}^{(l)}(t-h)U_{jq}^{(l)}(h)|^2dh.\tag{3.78}$$

Using (3.78), the fact that $|U_{jk}^{(l)}(t)|\le 1$, and the inequality $2ab\le a^2+b^2$, it follows that

$$\begin{aligned}
&\left|D_{pq}\{U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)\}\right|^2\\
&\le 2|D_{pq}U_{jj}^{(l)}(t_2)|^2+2|D_{pq}U_{jj}^{(l)}(t_1-t_2)|^2\\
&\le 8|t|\left(\int_0^{t_2}|U_{jp}^{(l)}(t_2-h)U_{jq}^{(l)}(h)|^2dh+\int_0^{t_1-t_2}|U_{jp}^{(l)}(t_1-t_2-h)U_{jq}^{(l)}(h)|^2dh\right).
\end{aligned}\tag{3.79}$$

Using the Poincaré inequality, (3.79), adding more nonnegative terms, and using the property of the unitary matrices that

$$\sum_{k=1}^{n}|U_{jk}^{(l)}(t)|^2=1,\tag{3.80}$$

it follows that

$$
\begin{aligned}
&\mathbf{Var}\left\{U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)\right\} \\
\leq\ & \sum_{\substack{p \leq q \\ p,q \in B_l}} \mathbb{E}\left[(M_{pq}^{(l)})^2\right]\mathbb{E}\left[\left|D_{pq}\{U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)\}\right|^2\right] \\
\leq\ & \frac{8(\sigma^2+1)|t|}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\mathbb{E}\left[\int_0^{t_2}\left|U_{jp}^{(l)}(t_2-h)U_{jq}^{(l)}(h)\right|^2 dh + \int_0^{t_1-t_2}\left|U_{jp}^{(l)}(t_1-t_2-h)U_{jq}^{(l)}(h)\right|^2 dh\right] \\
\leq\ & \frac{8(\sigma^2+1)|t|}{n}\sum_{p=1}^{n}\mathbb{E}\left[\int_0^{t_2}\left|U_{jp}^{(l)}(t_2-h)\right|^2 dh + \int_0^{t_1-t_2}\left|U_{jp}^{(l)}(t_1-t_2-h)\right|^2 dh\right] \\
\leq\ & \frac{16(\sigma^2+1)|t|}{n}t_1 \\
=\ & O\left(\frac{1}{n}\right).
\end{aligned}
\tag{3.81}
$$

Now, combining (3.76) with (3.81), we have that

$$
\mathbb{E}\left[\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{\{j\in B_l\}}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)e_n^{\circ}(x)\right] = O\left(\frac{1}{n}\right), \tag{3.82}
$$

and it follows that

$$
\left|-\frac{(\sigma^2-2)}{n}\int_0^t\int_0^{t_1}\mathbb{E}\left[\sum_{j=1}^{n}\mathbf{1}_{\{j\in B_l\}}U_{jj}^{(l)}(t_2)U_{jj}^{(l)}(t_1-t_2)e_n^{\circ}(x)\right]dt_2dt_1\right| = O\left(\frac{1}{n}\right). \tag{3.83}
$$

Now consider the final term of the remainder, given by (3.52). We apply the identity below

$$
\varphi_r'(M^{(r)}) = i\int_{-\infty}^{\infty}h\widehat{\varphi}_r(h)U^{(r)}(h)dh, \tag{3.84}
$$

which is a consequence of the matrix version of the Fourier inversion formula (3.21). Using (3.84), the finiteness of the integral (3.33), the above estimate (3.82), and the dominated convergence theorem, we have that

$$
\begin{aligned}
&\left|-\frac{x(\sigma^2-2)}{n}\sum_{r=1}^{d}\alpha_r\int_0^t\sum_{j=1}^{n}\mathbf{1}_{\{j\in B_l\cap B_r\}}\mathbb{E}\left[U_{jj}^{(l)}(t_1)(\varphi_r'(M^{(r)}))_{jj}e_n^{\circ}(x)\right]dt_1\right| \\
\leq\ & \sum_{r=1}^{d}|x(\sigma^2-2)\alpha_r|\left|\int_0^t\int_{-\infty}^{\infty}h\widehat{\varphi}_r(h)\frac{1}{n}\sum_{j\in B_l\cap B_r}\mathbb{E}\left[U_{jj}^{(l)}(t_1)U_{jj}^{(r)}(h)_{jj}e_n^{\circ}(x)\right]dhdt_1\right| \to 0.
\end{aligned}
\tag{3.85}
$$

Combining (3.69), (3.70), (3.75), (3.83), (3.85), and comparing to the remainder term (3.48), the proposition is proved. □

The goal now is to pass to the limit in (3.45). In what follows let $\{U_k^\gamma(x)\}$ denote the (rescaled) Chebyshev polynomials of the second kind on $[-2\sqrt{\gamma},\ 2\sqrt{\gamma}]$,

$$U_k^\gamma(x) = \sum_{j=0}^{\lfloor k/2 \rfloor} (-1)^j \binom{k-j}{j} \left(\frac{x}{2\sqrt{\gamma}}\right)^{k-2j}. \tag{3.86}$$

Proposition 3.4. *Let $A_n^{(l)}(t)$ be given by (3.46), $Q_n^{(l)}(t)$ given by (3.47), and $\bar{v}_n(t)$ given by (3.44) . Then the limits of $A_n^{(l)}(t), Q_n^{(l)}(t)$ and $\bar{v}_n(t)$ as $n \to \infty$ exist and*

$$\begin{aligned} A^{(l)}(t) &:= \lim_{n\to\infty} A_n^{(l)}(t) \\ &= -\frac{1}{2\pi^2\gamma_l} \sum_{r=1}^{d} \frac{\alpha_r}{\gamma_r} \int_0^t \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} e^{it_1 x} \varphi_r'(y) \sqrt{4\gamma_l - x^2} \sqrt{4\gamma_r - y^2} F_{lr}(x,y) dy dx dt_1, \end{aligned} \tag{3.87}$$

where

$$F_{lr}(x,y) = \sum_{k=0}^{\infty} U_k^{\gamma_l}(x) U_k^{\gamma_r}(y) \frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2}\gamma_r^{k/2}}, \tag{3.88}$$

the limit of $Q_n^{(l)}(t)$ is given by

$$\begin{aligned} Q^{(l)}(t) &:= \lim_{n\to\infty} Q_n^{(l)}(t) \\ &= -\frac{(\sigma^2-2)}{4\pi^2\gamma_l} \sum_{r=1}^{d} \frac{\gamma_{lr}\alpha_r}{\gamma_r} \int_0^t \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} e^{it_1\lambda} \sqrt{4\gamma_l - \lambda^2} d\lambda dt_1 \cdot \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} \varphi_r'(\mu) \sqrt{4\gamma_r - \mu^2} d\mu, \end{aligned} \tag{3.89}$$

and the limit of $\bar{v}_n(t)$, after rescaling by γ_l, is given by

$$v^{(l)}(t) := \frac{1}{\gamma_l} \lim_{n\to\infty} \bar{v}_n(t) = \frac{1}{2\pi\gamma_l} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} e^{itx} \sqrt{4\gamma_l - x^2} dx. \tag{3.90}$$

Proof. Recall that $A_n^{(l)}(t) = -2\sum_{r=1}^d \alpha_r \int_0^t \frac{1}{n} \mathbb{E}\left[\mathrm{Tr}\{P^{(l)} U^{(l)}(t_1) P^{(l,r)} \varphi_r'(M^{(r)}) P^{(r)}\}\right] dt_1$. In the full Wigner matrix case one has $A_n(t) = -2\int_0^t \frac{1}{n} \mathbb{E}\mathrm{Tr}\{e^{itM}\varphi'(M)\} dt_1$, and the limiting behavior follows immediately from the Wigner semicircle law. In the case of submatrices with asymptotically regular intersections there are additional technical difficulties due to the fact that for the $n \times n$ submatrices $M^{(l)} = P^{(l)} M P^{(l)}$, we have

$$\mathrm{Tr}\{P^{(l,r)} U^{(l)}(t) \varphi_r'(M^{(r)}) P^{(l,r)}\} = \sum_{j,k \in B_l \cap B_r} U_{jk}^{(l)}(t) \varphi_r'(M^{(r)})_{jk}, \tag{3.91}$$

so that the summation is restricted to entries common to both submatrices, i.e. to $j, k \in B_l \cap B_r$. It follows from lemma 2.5 that the limit of $A_n^{(l)}(t)$ exists and equals

$$A^{(l)}(t) = -2\sum_{r=1}^{d} \alpha_r \int_0^t \langle e^{it_1 x}, \varphi_r' \rangle_{lr} \, dt_1, \tag{3.92}$$

where

$$\langle e^{it_1x},\ \varphi_r'\rangle_{lr} \ = \frac{1}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} e^{it_1x}\varphi_r'(y)F_{lr}(x,y)\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}dydx. \tag{3.93}$$

This establishes (3.87). The proof of lemma 2.5 will be given in section 3.2.

We turn our attention to $Q_n^{(l)}(t)$. First it will be argued that the variance of the matrix entries converge to zero. Using the Poincaré inequality, (3.78), (3.80), and proposition 3.1, it follows that

$$\begin{aligned}
&\mathbf{Var}\left\{U_{jj}^{(l)}(t_1)\right\} \\
&\leq \sum_{p\leq q,p,q\in B_l}\mathbb{E}\left[(M_{pq}^{(l)})^2\right]\mathbb{E}\left[\left|D_{pq}U_{jj}^{(l)}(t)\right|^2\right] \\
&\leq \frac{4(\sigma^2+1)|t_1|}{n}\sum_{p=1}^{n}\sum_{q=1}^{n}\mathbb{E}\int_0^{t_1}\left|U_{jp}^{(l)}(t_1-t_2)U_{jq}^{(l)}(t_2)\right|^2dt_2 \\
&\leq \frac{4(\sigma^2+1)|t_1|}{n}\sum_{p=1}^{n}\mathbb{E}\int_0^{t_1}\left|U_{jp}^{(l)}(t_1-t_2)\right|^2dt_2 \\
&\leq \frac{4(\sigma^2+1)t_1^2}{n} = O\left(n^{-1}\right).
\end{aligned} \tag{3.94}$$

Note that in the course of the calculation (3.94), we showed that

$$\sum_{p\leq q}\mathbb{E}\left[\left|D_{pq}U_{jj}^{(l)}(t_1)\right|^2\right]\leq 4t_1^2. \tag{3.95}$$

The Cauchy-Schwartz inequality implies

$$\int_{\mathbb{R}}(1+t_1^2)|\widehat{\varphi}_r(t_1)|dt_1 \leq \sqrt{\int_{\mathbb{R}}\frac{dt_1}{(1+t_1^2)^{1/2+\epsilon}}}\cdot\sqrt{\int_{\mathbb{R}}(1+t_1^2)^{5/2+\epsilon}|\widehat{\varphi}_r(t_1)|^2dt_1}. \tag{3.96}$$

Since $||\varphi_r||_{5/2+\epsilon}<\infty$, we have the estimate

$$\int_{-\infty}^{\infty}t_1^2|\widehat{\varphi}_r(t_1)|dt_1<\infty. \tag{3.97}$$

Using the Cauchy-Schwartz inequality and (3.84), it follows that

$$\begin{aligned}
\left|D_{pq}\varphi_r'(M^{(r)})_{jj}\right|^2 &= \left|\int_{-\infty}^{\infty}t_1\widehat{\varphi}_r(t_1)D_{pq}U_{jj}^{(l)}(t_1)dt_1\right|^2 \\
&\leq \int_{-\infty}^{\infty}t_1^2|\widehat{\varphi}_r(t_1)|dt_1\cdot\int_{-\infty}^{\infty}|\widehat{\varphi}_r(t_1)|\cdot\left|D_{pq}U_{jj}^{(l)}(t_1)\right|^2dt_1.
\end{aligned} \tag{3.98}$$

Using the Poincaré inequality, (3.95), (3.98), we obtain

$$\begin{aligned}
&\mathbf{Var}\left\{\varphi_r'(M^{(r)})_{jj}\right\} \\
&\le \sum_{p\le q}\mathbb{E}\left[(M_{pq}^{(l)})^2\right]\mathbb{E}\left[\left|D_{pq}\varphi_r'(M^{(r)})_{jj}\right|^2\right] \\
&\le \frac{(\sigma^2+1)}{n}\cdot\int_{-\infty}^{\infty} t_1^2|\widehat{\varphi}_r(t_1)|dt_1\cdot\sum_{p\le q}\int_{-\infty}^{\infty}|\widehat{\varphi}_r(t_1)|\mathbb{E}\left[\left|D_{pq}U_{jj}^{(l)}(t_1)\right|^2\right]dt_1 \\
&\le \frac{4(\sigma^2+1)}{n}\cdot\left(\int_{-\infty}^{\infty} t_1^2|\widehat{\varphi}_r(t_1)|dt_1\right)^2. \qquad (3.99)
\end{aligned}$$

Using (3.97), (3.99), (3.94), and the Cauchy-Schwartz inequality, we obtain

$$\mathbf{Cov}\{U_{jj}^{(l)}(t_1),\varphi_r'(M^{(r)})_{jj}\}\le\sqrt{\mathbf{Var}\{U_{jj}^{(l)}(t_1)\}}\cdot\sqrt{\mathbf{Var}\left\{\varphi_r'(M^{(r)})_{jj}\right\}}=O\left(n^{-1}\right). \qquad (3.100)$$

Using (3.100) it is justified to replace the expectation $\mathbb{E}[U_{jj}^{(l)}(t)\varphi_r'(M^{(r)})_{jj}]$ by the product $\mathbb{E}[U_{jj}^{(l)}(t)]\cdot \mathbb{E}[\varphi_r'(M^{(r)})_{jj}]$, when passing to the limit. We use proposition 2.1 of [15], which guarantees that for $f\in C_c^7(\mathbb{R})$,

$$\lim_{n\to\infty}\mathbb{E}[f(M)_{jj}]=\int_{\mathbb{R}} f(x)d\mu_{sc}(x). \qquad (3.101)$$

In order to apply this asymptotic to the exponential function, which is smooth enough, we truncate the function in a smooth fashion outside the support of μ_{sc}. We are justified in replacing the exponential function by its truncated version because the eigenvalues of the submatrices concentrate in the support of the semicircle law, with overwhelming probability. It is for this same reason that we may assume φ_r' is compactly supported. This function is not sufficiently smooth, but we can avoid this problem by a density argument using standard convolution, and then apply the bound (3.3) on the variance of linear eigenvalue statistics.

Let $\eta\in C_c^\infty(\mathbb{R})$ satisfy $\int_{\mathbb{R}}\eta(x)dx=1$, and consider the mollifiers $\eta_y(x):=y^{-1}\eta(xy^{-1})$. Then $\varphi_r'*\eta_y\in C_c^\infty(\mathbb{R})$, and using standard Fourier theory it can be shown that

$$\lim_{y\to 0}||\varphi_r'-\varphi_r'*\eta_y||_{3/2+\epsilon}^2=0. \qquad (3.102)$$

It follows from (3.100) and (3.101) that

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\mathbf{1}_{\{j\in B_l\cap B_r\}}\mathbb{E}\left[U_{jj}^{(l)}(t)\varphi_r'(M^{(r)})_{jj}\right]= \\
&\gamma_{lr}\left(\frac{1}{2\pi\gamma_l}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}e^{it\lambda}\sqrt{4\gamma_l-\lambda^2}d\lambda\right)\cdot\left(\frac{1}{2\pi\gamma_r}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\varphi_r'(\mu)\sqrt{4\gamma_r-\mu^2}d\mu\right). \qquad (3.103)
\end{aligned}$$

Using (3.103), we pass to the limit in (3.47), and obtain (3.89). The limit of

$$\bar{v}_n^{(l)}(t)=\frac{1}{n}\mathbb{E}[u_n^{(l)}(t)]\approx\frac{\gamma_l}{|B_l|}\mathbb{E}[\mathrm{Tr}\{P^{(l)}U^{(l)}(t)P^{(l)}\}],$$

is given by (rescaled) Wigner semicircle law, as a consequence of the zero eigenvalues. Alternatively, it can be computed using the bilinear form in lemma 2.5, with $f(x) = e^{itx}$ and $g(x) = 1$. To facilitate solving the integral equation (3.105), below, it will be useful to rescale by γ_l. We obtain

$$\begin{aligned} v^{(l)}(t) &= \frac{1}{\gamma_l}\langle e^{itx}, 1\rangle_{ll} \\ &= \frac{1}{2\pi\gamma_l}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} e^{itx}\sqrt{4\gamma_l - x^2}dx, \end{aligned} \tag{3.104}$$

which establishes (3.90). The proposition is proved. □

Now using propositions 3.2, 3.3, 3.4, we pass to the limit $n_m \to \infty$ in (3.45), and determine that the limit $Y^{(l)}$ of every uniformly converging subsequence $\{Y^{(l)}_{n_m}\}$ satisfies the equation

$$Y^l(x,t) + 2\gamma_l\int_0^t\int_0^{t_1} v^{(l)}(t_1 - t_2)Y^{(l)}(x,t_2)dt_2dt_1 = xZ(x)\left[A^{(l)}(t) + Q^{(l)}(t)\right], \tag{3.105}$$

where $A^{(l)}(t)$ is given by (3.87), $Q^{(l)}(t)$ is given by (3.89), and $v^{(l)}(t)$ is given by (3.90).

Now the argument will proceed by solving the integral equation (3.105). We use a version of the technique used by L. Pastur and A. Lytova in [10], to solve this equation. Define

$$f(z) := (\sqrt{z^2 - 4\gamma_l} - z)/2\gamma_l, \tag{3.106}$$

which is the Stieltjes transform of the rescaled semicircle law, where $\sqrt{z^2 - 4\gamma_l} = z + O(1/z)$ as $z \to \infty$. A direct calculation shows that $\tilde{v}^{(l)} = f$, where $\tilde{v}^{(l)}$ denotes the generalized Fourier transform of $v^{(l)}$. We obtain

$$\begin{aligned} \tilde{v}^{(l)}(z) &:= \frac{1}{2\pi i\gamma_l}\int_0^\infty\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} e^{it(x-z)}\sqrt{4\gamma_l - x^2}dxdt \\ &= \frac{1}{2\pi\gamma_l}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \frac{1}{x-z}\sqrt{4\gamma_l - x^2}dx \\ &= f(z). \end{aligned} \tag{3.107}$$

We check that

$$z + 2\gamma_l f(z) = \sqrt{z^2 - 4\gamma_l} \neq 0, \quad \mathfrak{Im}\, z \neq 0. \tag{3.108}$$

Set

$$T(t) := \frac{i}{2\pi}\int_L \frac{e^{izt}dz}{z + 2\gamma_l f(z)} = -\frac{1}{\pi}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \frac{e^{i\lambda t}d\lambda}{\sqrt{4\gamma_l - \lambda^2}}, \tag{3.109}$$

after replacing the integral over L by the integral over $[-2\gamma_l, 2\gamma_l]$, and taking into account that $\sqrt{z^2 - 4\gamma_l}$ is $\pm i\sqrt{4\gamma_l - \lambda^2}$, on the upper and lower edges of the cut. Then the solution of (3.105) is

$$Y^{(l)}(x,t) = -xZ(x)\int_0^t T(t - t_1)\frac{d}{dt_1}\left[A^{(l)}(t_1) + Q^{(l)}(t_1)\right]dt_1. \tag{3.110}$$

Then, with F_{lr} given by (3.88),

$$
\begin{aligned}
&\int_0^t T(t-t_1)\frac{d}{dt_1}A^{(l)}(t_1)dt_1 \\
&= \frac{1}{2\pi^3\gamma_l}\sum_{r=1}^d \frac{\alpha_r}{\gamma_r}\int_0^t \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} e^{i(t-t_1)\lambda}e^{it_1x}\varphi_r'(y)\frac{\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}}{\sqrt{4\gamma_l-\lambda^2}} \\
&\quad \times F_{lr}(x,y)dydxd\lambda dt_1 \\
&= \frac{1}{2i\pi^3\gamma_l}\sum_{r=1}^d \frac{\alpha_r}{\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} \frac{\left[e^{itx}-e^{it\lambda}\right]\varphi_r'(y)}{(x-\lambda)}\frac{\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}}{\sqrt{4\gamma_l-\lambda^2}}F_{lr}(x,y)dydxd\lambda,
\end{aligned}
\tag{3.111}
$$

and

$$
\begin{aligned}
&\int_0^t T(t-t_1)\frac{d}{dt_1}Q^{(l)}(t_1)dt_1 \\
&= -\frac{\gamma_{lr}(\sigma^2-2)}{4\pi^3\gamma_l}\sum_{r=1}^d \frac{\alpha_r}{\gamma_r}\int_0^t \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \frac{e^{i(t-t_1)\lambda}}{\sqrt{4\gamma_l-\lambda^2}}e^{it_1\eta}\sqrt{4\gamma_l-\eta^2}d\eta d\lambda \\
&\quad \times \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} \varphi_r'(\mu)\sqrt{4\gamma_r-\mu^2}d\mu dt_1 \\
&= -\frac{\gamma_{lr}(\sigma^2-2)}{4\pi^3\gamma_l i}\sum_{r=1}^d \frac{\alpha_r}{\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\left[\frac{e^{it\eta}-e^{it\lambda}}{\eta-\lambda}\right]\frac{\sqrt{4\gamma_l-\eta^2}}{\sqrt{4\gamma_l-\lambda^2}}d\eta d\lambda\cdot\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\varphi_r'(\mu)\sqrt{4\gamma_r-\mu^2}d\mu.
\end{aligned}
\tag{3.112}
$$

Using the regularity condition $||\varphi_l||_{5/2+\epsilon}<\infty$ for $1\le l\le d$, (3.111), (3.112), and the dominated convergence theorem to pass to limit in (3.24) yields

$$
\begin{aligned}
&Z'(x) \\
&= i\sum_{l=1}^d \alpha_l\int_{-\infty}^{\infty}\widehat{\varphi}_l(t)Y^{(l)}(x,t)dt \\
&= -\frac{xZ(x)}{2\pi^3}\sum_{l=1}^d\sum_{r=1}^d \frac{\alpha_l\alpha_r}{\gamma_l\gamma_r}\int_{-\infty}^{\infty}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\widehat{\varphi}_l(t)\frac{\left[e^{itx}-e^{it\lambda}\right]\varphi_r'(y)}{(x-\lambda)} \\
&\quad \times\frac{\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}}{\sqrt{4\gamma_l-\lambda^2}}F_{lr}(x,y)dydxd\lambda dt \\
&\quad -\frac{\gamma_{lr}(\sigma^2-2)xZ(x)}{4\pi^3}\sum_{l=1}^d\sum_{r=1}^d \frac{\alpha_l\alpha_r}{\gamma_l\gamma_r}\int_{-\infty}^{\infty}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\left[\frac{\widehat{\varphi}_l(t)e^{it\eta}-\widehat{\varphi}_l(t)e^{it\lambda}}{\eta-\lambda}\right]\frac{\sqrt{4\gamma_l-\eta^2}}{\sqrt{4\gamma_l-\lambda^2}}d\eta d\lambda \\
&\quad \times\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\varphi_r'(\mu)\sqrt{4\gamma_r-\mu^2}d\mu dt.
\end{aligned}
\tag{3.113}
$$

Applying the Fourier inversion formula (3.21), it follows that

$$
\begin{aligned}
&Z'(x) = \\
&-\frac{xZ(x)}{2\pi^3}\sum_{l=1}^{d}\sum_{r=1}^{d}\frac{\alpha_l\alpha_r}{\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\frac{[\varphi_l(x)-\varphi_l(\lambda)]\,\varphi_r'(y)}{(x-\lambda)}\frac{\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}}{\sqrt{4\gamma_l-\lambda^2}} \\
&\times F_{lr}(x,y)dydxd\lambda \\
&-xZ(x)\frac{\gamma_{lr}(\sigma^2-2)}{4\pi^3}\sum_{l=1}^{d}\sum_{r=1}^{d}\frac{\alpha_l\alpha_r}{\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\left[\frac{\varphi_l(\eta)-\varphi_l(\lambda)}{\eta-\lambda}\right]\frac{\sqrt{4\gamma_l-\eta^2}}{\sqrt{4\gamma_l-\lambda^2}}d\eta d\lambda \\
&\times\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\varphi_r'(\mu)\sqrt{4\gamma_r-\mu^2}d\mu. \qquad (3.114)
\end{aligned}
$$

We will use the fact that

$$\int_{-2\sqrt{\gamma}}^{2\sqrt{\gamma}}\frac{\left[T_k^{\gamma}(x)-T_k^{\gamma}(\lambda)\right]}{(x-\lambda)}\frac{d\lambda}{\sqrt{4\gamma-\lambda^2}} = \frac{\pi}{2\sqrt{\gamma}}U_{k-1}^{\gamma}(x), \quad k\geq 1. \qquad (3.115)$$

Expand the test function φ_l in the Chebyshev basis to obtain

$$\varphi_l(x)=\sum_{k=0}^{\infty}(\varphi_l)_k T_k^{\gamma_l}(x), \quad (\varphi_l)_k=\frac{2}{\pi}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\varphi_l(t)T_k^{\gamma_l}(t)\frac{dt}{\sqrt{4\gamma_l-t^2}}. \qquad (3.116)$$

Returning to the computation of $Z'(x)$, using (3.114), (3.115), and (3.116), it follows that

$$
\begin{aligned}
Z'(x) \;=\; & -\frac{xZ(x)}{4\pi^2}\sum_{l=1}^{d}\sum_{r=1}^{d}\sum_{k=1}^{\infty}\frac{\alpha_l\alpha_r}{\gamma_l^{3/2}\gamma_r}(\varphi_l)_k\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}U_{k-1}^{\gamma_l}(x)\varphi_r'(y)\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2} \\
&\times F_{lr}(x,y)dydx \\
&-xZ(x)\frac{\gamma_{lr}(\sigma^2-2)}{8\pi^2}\sum_{l=1}^{d}\sum_{r=1}^{d}\frac{\alpha_l\alpha_r}{\gamma_l^{3/2}\gamma_r}\sum_{k=1}^{\infty}(\varphi_l)_k\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}U_{k-1}^{\gamma_l}(\eta)\sqrt{4\gamma_l-\eta^2}d\eta \\
&\times\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\varphi_r'(\mu)\sqrt{4\gamma_r-\mu^2}d\mu. \qquad (3.117)
\end{aligned}
$$

Using the orthogonality of the Chebyshev polynomials (2.21),

$$\sum_{k=1}^{\infty}(\varphi_l)_k\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}U_{k-1}^{\gamma_l}(\eta)\sqrt{4\gamma_l-\eta^2}d\eta = 2\sqrt{\gamma_l}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\frac{\lambda\varphi_l(\lambda)}{\sqrt{4\gamma_l-\lambda^2}}d\lambda. \qquad (3.118)$$

Integrating by parts yields

$$\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\varphi_r'(\mu)\sqrt{4\gamma_r-\mu^2}d\mu = \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\frac{\mu\varphi_r(\mu)}{\sqrt{4\gamma_r-\mu^2}}d\mu, \qquad (3.119)$$

so that

$$\frac{\gamma_{lr}(\sigma^2-2)}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\frac{\lambda\varphi_l(\lambda)}{\sqrt{4\gamma_l-\lambda^2}}d\lambda\cdot\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\frac{\mu\varphi_r(\mu)}{\sqrt{4\gamma_r-\mu^2}}d\mu = \frac{(\sigma^2-2)}{4}\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}}(\varphi_l)_1(\varphi_r)_1. \qquad (3.120)$$

Since

$$\frac{d}{dy}T_k^{\gamma}(y) = \frac{k}{2\sqrt{\gamma}}U_{k-1}^{\gamma}(y), \tag{3.121}$$

we expand $\varphi_r(y)$ in the Chebyshev basis to obtain

$$\varphi_r'(y) = \frac{1}{2\sqrt{\gamma_r}}\sum_{m=1}^{\infty} m(\varphi_r)_m U_{m-1}^{\gamma_r}(y). \tag{3.122}$$

Recalling that F_{lr} is given by (3.88), it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{m=1}^{\infty} m(\varphi_l)_k(\varphi_r)_m \left[\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} U_{k-1}^{\gamma_l}(x)U_{m-1}^{\gamma_r}(y)\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}F_{lr}(x,y)dydx\right]\\
=\;& \sum_{k=1}^{\infty}\sum_{m=1}^{\infty}\sum_{j=0}^{\infty} m(\varphi_l)_k(\varphi_r)_m \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} U_{k-1}^{\gamma_l}(x)U_{m-1}^{\gamma_r}(y)U_j^{\gamma_l}(x)U_j^{\gamma_r}(y)\\
&\times\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}dydx\frac{\gamma_{lr}^{j+1}}{\gamma_l^{j/2}\gamma_r^{j/2}}.\\
=\;& 4\pi^2\gamma_l\gamma_r\sum_{k=1}^{\infty} k(\varphi_l)_k(\varphi_r)_k\left(\frac{\gamma_{lr}^k}{\gamma_l^{(k-1)/2}\gamma_r^{(k-1)/2)}}\right).
\end{aligned} \tag{3.123}$$

Using (3.123), (3.118), (3.119) and (3.120), in (3.117), it follows that

$$\begin{aligned}
Z'(x) \;=\;& -\frac{xZ(x)}{2}\sum_{l=1}^{d}\sum_{r=1}^{d}\alpha_l\alpha_r\left[\frac{(\sigma^2-2)}{2}\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}}(\varphi_l)_1(\varphi_r)_1+\sum_{k=1}^{\infty}k(\varphi_l)_k(\varphi_r)_k\left(\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}}\right)^k\right]\\
=\;& -xZ(x)\sum_{l=1}^{d}\alpha_l^2\left[\frac{\sigma^2}{4}(\varphi_l)_1^2+\frac{1}{2}\sum_{k=2}^{\infty}k(\varphi_l)_k^2\right]\\
& -xZ(x)\sum_{1\le l<r\le d}2\alpha_l\alpha_r\left[\frac{\sigma^2}{4}(\varphi_l)_1(\varphi_r)_1(\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}})+\frac{1}{2}\sum_{k=2}^{\infty}k(\varphi_l)_k(\varphi_r)_k\left(\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}}\right)^k\right].
\end{aligned} \tag{3.124}$$

We have obtained the expression for the asymptotic covariance (2.13) in terms of Chebyshev polynomials. Now we write this expression as a contour integral. Let

$$\beta := \frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}},$$

make the change of coordinates $x = 2\sqrt{\gamma_l}cos(\theta),\ y = 2\sqrt{\gamma_r}cos(\omega)$, and use (2.14) to obtain that

$$\begin{aligned}
&\frac{1}{2}\sum_{k=1}^{\infty}k\beta^k(\varphi_l)_k(\varphi_r)_k\\
=\;& \frac{2}{\pi^2}\sum_{k=1}^{\infty}k\beta^k\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\varphi_l(x)\varphi_r(y)T_k\left(\frac{x}{2\sqrt{\gamma_l}}\right)T_k\left(\frac{y}{2\sqrt{\gamma_r}}\right)\frac{dxdy}{\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}}\\
=\;& \frac{2}{\pi^2}\int_0^{\pi}\int_0^{\pi}\sum_{k=1}^{\infty}k\beta^k cos(k\theta)cos(k\omega)\varphi_l\left(2\sqrt{\gamma_l}cos\theta\right)\varphi_r\left(2\sqrt{\gamma_r}cos\omega\right)d\theta d\omega.
\end{aligned} \tag{3.125}$$

Integrating by parts in θ, ω it follows that

$$\frac{1}{2}\sum_{k=1}^{\infty} k\beta^k (\varphi_l)_k(\varphi_r)_k = \frac{2}{\pi^2}\int_0^{\pi}\int_0^{\pi} \varphi_l'\left(2\sqrt{\gamma_l}cos\theta\right)\varphi_r'\left(2\sqrt{\gamma_r}cos\omega\right)\left[\sum_{k=1}^{\infty}\frac{\beta^k}{k}sin(k\theta)sin(k\omega)\right] \times(2\sqrt{\gamma_l}sin\theta)(2\sqrt{\gamma_r}sin\omega)d\theta d\omega. \tag{3.126}$$

To evaluate the infinite sum above, recall that for $z \in \mathbb{C}$ with $|z| < 1$, we have

$$\ln(1-z) = -\sum_{k=1}^{\infty}\frac{z^k}{k}. \tag{3.127}$$

Noting that $\beta < 1$, using (3.127), it follows that

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{\beta^k}{k}sin(k\theta)sin(k\omega)\\
&= -\frac{1}{4}\sum_{k=1}^{\infty}\frac{\beta^k}{k}\left[e^{ik\theta}-e^{-ik\theta}\right]\left[e^{ik\omega}-e^{-ik\omega}\right]\\
&= -\frac{1}{4}\left[-\ln\left(1-\beta e^{i(\theta+\omega)}\right)+\ln\left(1-\beta e^{i(\theta-\omega)}\right)-\ln\left(1-\beta e^{-i(\theta+\omega)}\right)+\ln\left(1-\beta e^{-i(\theta-\omega)}\right)\right]\\
&= -\frac{1}{4}\left[\ln\left[\left(1-\beta e^{i(\theta-\omega)}\right)\overline{\left(1-\beta e^{i(\theta-\omega)}\right)}\right]-\ln\left[\left(1-\beta e^{i(\theta+\omega)}\right)\overline{\left(1-\beta e^{i(\theta+\omega)}\right)}\right]\right].
\end{aligned} \tag{3.128}$$

Making the change of coordinates $z = \sqrt{\gamma_l}e^{i\theta}$, $w = \sqrt{\gamma_r}e^{i\omega}$, and recalling that $\beta = \frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}}$, this can be written as

$$\begin{aligned}
\sum_{k=1}^{\infty}\frac{\beta^k}{k}\sin(k\theta)\sin(k\omega) &= -\frac{1}{4}\ln\left[\frac{\left(1-\beta e^{i(\theta-\omega)}\right)\overline{\left(1-\beta e^{i(\theta-\omega)}\right)}}{\left(1-\beta e^{i(\theta+\omega)}\right)\overline{\left(1-\beta e^{i(\theta+\omega)}\right)}}\right]\\
&= -\frac{1}{4}\ln\left[\frac{\left(1-\frac{\gamma_{lr}}{\gamma_l\gamma_r}z\overline{w}\right)\overline{\left(1-\frac{\gamma_{lr}}{\gamma_l\gamma_r}z\overline{w}\right)}}{\left(1-\frac{\gamma_{lr}}{\gamma_l\gamma_r}zw\right)\overline{\left(1-\frac{\gamma_{lr}}{\gamma_l\gamma_r}zw\right)}}\right]\\
&= -\frac{1}{4}\ln\left[\frac{|\gamma_{lr}-z\overline{w}|^2}{|\gamma_{lr}-zw|^2}\right]\\
&= \frac{1}{2}\ln\left|\frac{\gamma_{lr}-zw}{\gamma_{lr}-z\overline{w}}\right|.
\end{aligned} \tag{3.129}$$

Combining (3.126), (3.129), and noting that

$$(2\sqrt{\gamma_l}\sin\theta)(2\sqrt{\gamma_r}\sin\omega)\,d\theta d\omega = \left(1-\frac{\gamma_l}{z^2}\right)\left(1-\frac{\gamma_r}{w^2}\right)dzdw,$$

it follows that

$$\frac{1}{2}\sum_{k=1}^{\infty}k\beta^k(\varphi_l)_k(\varphi_r)_k = \frac{2}{\pi}\oint\limits_{\substack{|z|^2=\gamma_l\\ \mathfrak{Im}\,z>0}}\ \oint\limits_{\substack{|w|^2=\gamma_r\\ \mathfrak{Im}\,w>0}} \varphi_l'\left(z+\frac{\gamma_l}{z}\right)\varphi_r'\left(w+\frac{\gamma_r}{w}\right)\frac{1}{2\pi}\ln\left|\frac{\gamma_{lr}-zw}{\gamma_{lr}-z\bar{w}}\right|\left(1-\frac{\gamma_l}{z^2}\right)\left(1-\frac{\gamma_r}{w^2}\right)dzdw. \tag{3.130}$$

Compare (3.124) to (3.8). Using (3.130), (3.13), (3.14) and (3.9), it follows that the covariance can be written as

$$
\begin{aligned}
&\lim_{n\to\infty} \mathbf{Cov}\{\mathcal{N}^{(l)}[\varphi_l],\ \mathcal{N}^{(r)}[\varphi_r]\} \\
&= \frac{\sigma^2}{4}(\varphi_l)_1(\varphi_r)_1\left(\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}}\right) + \frac{1}{2}\sum_{k=2}^{\infty} k(\varphi_l)_k(\varphi_r)_k\left(\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}}\right)^k \\
&= \frac{2}{\pi}\oint_{\substack{|z|^2=\gamma_l \\ \mathfrak{Im}\ z>0}}\ \oint_{\substack{|w|^2=\gamma_r \\ \mathfrak{Im}\ w>0}} \varphi_l'\left(z+\frac{\gamma_l}{z}\right)\varphi_r'\left(w+\frac{\gamma_r}{w}\right)\frac{1}{2\pi}log\left|\frac{\gamma_{lr}-zw}{\gamma_{lr}-z\bar{w}}\right|\left(1-\frac{\gamma_l}{z^2}\right) \\
&\quad\times\left(1-\frac{\gamma_r}{w^2}\right)dwdz + \frac{\gamma_{lr}(\sigma^2-2)}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\frac{\lambda\varphi_l(\lambda)}{\sqrt{4\gamma_l-\lambda^2}}d\lambda\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}}\frac{\mu\varphi_r(\mu)}{\sqrt{4\gamma_r-\mu^2}}d\mu.
\end{aligned}
$$

3.2 The Bilinear Form

The main goal of this section is to prove lemma 2.5, to which we now turn our attention. Begin with the following definition.

Definition 3.5. Let M be a Wigner matrix satisfying (1.1), and let $P^{(l)}$, $P^{(l,r)}$ be the projection matrices defined in (2.6) and (2.10). For polynomial functions $f, g : \mathbb{R} \to \mathbb{R}$, define

$$
\begin{aligned}
\langle f, g\rangle_{lr,n} &:= \frac{1}{n}\sum_{j,k\in B_l\cap B_r}\mathbb{E}\left[f(M^{(l)})_{jk}\cdot g(M^{(r)})_{kj}\right] \\
&= \frac{1}{n}\mathbb{E}\left[\mathrm{Tr}\left\{P^{(l)}f(M^{(l)})\cdot P^{(l,r)}\cdot g(M^{(r)})P^{(r)}\right\}\right]. \qquad (3.131)
\end{aligned}
$$

The large n limit of $\langle f, g\rangle_{lr,n}$ exists for polynomial functions because all moments of the matrix entries of M are finite. Then $\lim_{n\to\infty}\langle f, g\rangle_{lr,n} = \langle f, g\rangle_{lr}$, where $\langle\cdot,\cdot\rangle_{lr}$ is the bilinear form defined in definition 2.3.

We compute the bilinear form $\langle f, g\rangle_{lr}$ of definition 2.3 for monomial functions $f(x) = x^k,\ g(x) = x^q$. We will also consider the random variables $n^{-1}\mathrm{Tr}\{P^{(l)}f(M^{(l)})P^{(l,r)}g(M^{(r)})P^{(r)}\}$, and prove their convergence almost surely to the non-random limit described in lemma 2.5. Some results and techniques from free probability theory will be used. See [1] for the relevant background concerning noncommutative probability spaces, asymptotic freeness of Wigner matrices and for the definition and properties of the multilinear free cumulant functionals κ_p, for $p \geq 1$. The notation is chosen to agree with that text.

Regard the matrices $M, P^{(l)}, P^{(r)}$ as noncommutative random variables in the noncommutative probability spaces $(Mat_n(\mathbb{C}), \mathbb{E}[\frac{1}{n}\mathrm{Tr}])$ and also $(Mat_n(\mathbb{C}), \frac{1}{n}\mathrm{Tr}\{\cdot\})$. Since M is a Wigner random matrix and $\{P^{(l)}, P^{(r)}\}$ are deterministic Hermitian matrices, it follows from part (i) of theorem 5.4.5 in [1] that M is asymptotically free from $\{P^{(l)}, P^{(r)}\}$ with respect to the functional $n^{-1}\mathbb{E}\mathrm{Tr}(\cdot)$. Also, it follows from part (ii) of theorem 5.4.5 in [1] that M is almost surely asymptotically free from $\{P^{(l)}, P^{(r)}\}$ with respect to the functional $n^{-1}\mathrm{Tr}(\cdot)$. The set of all non-crossing partitions over a set with p letters is denoted below by $NC(p)$. An important consequence of the asymptotic freeness and almost sure asymptotic freeness of these matrices is that mixed free cumulants of M and $\{P^{(l)}, P^{(r)}\}$ vanish in the limit, with respect to both functionals, see theorem 5.3.15 of [1].

Therefore, letting κ_π denote a product of free cumulant functionals corresponding to the block structure of the partition π, it follows that

$$
\begin{aligned}
\langle x^k,\ x^q\rangle_{lr} &= \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\mathrm{Tr}\left\{P^{(l)}\left(P^{(l)}MP^{(l)}\right)^k\left(P^{(r)}MP^{(r)}\right)^q P^{(r)}\right\}\right] \\
&= \lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left[\mathrm{Tr}\{P^{(l)}MP^{(l)}\cdots P^{(l)}MP^{(l)}P^{(r)}MP^{(r)}\cdots P^{(r)}MP^{(r)}\right] \\
&= \sum_{\pi\in NC(2(k+q)+1)} \kappa_\pi(P^{(l)}, M, P^{(l)}, \cdots, M, P^{(l)}, P^{(r)}, M, \cdots, M, P^{(r)}) \\
&= \sum_{\substack{\pi_1\in NC(odd),\, \pi_2\in NC(even)\\ \pi_1\cup\pi_2\in NC(2(k+q)+1)}} \kappa_{\pi_2}(M)\kappa_{\pi_1}(P^{(l)}, \cdots, P^{(l,r)}, \cdots, P^{(r)}),
\end{aligned}
\tag{3.132}
$$

and also that almost surely

$$
\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left\{P^{(l)}\left(P^{(l)}MP^{(l)}\right)^k\left(P^{(r)}MP^{(r)}\right)^q P^{(r)}\right\} \\
&= \sum_{\pi\in NC(2(k+q)+1)} \kappa_\pi(P^{(l)}, M, P^{(l)}, \cdots, M, P^{(l)}, P^{(r)}, M, \cdots, M, P^{(r)}) \\
&= \sum_{\substack{\pi_1\in NC(odd),\, \pi_2\in NC(even)\\ \pi_1\cup\pi_2\in NC(2(k+q)+1)}} \kappa_{\pi_2}(M)\ \kappa_{\pi_1}(P^{(l)}, \cdots, P^{(l,r)}, \cdots, P^{(r)}).
\end{aligned}
\tag{3.133}
$$

Above $NC(odd)$, for example, denotes the set of non-crossing partitions on the odd integers in the indicated set. Since the calculation of the joint moments in each non-commutative probability space $(Mat_n(\mathbb{C}), n^{-1}\mathbb{E}\mathrm{Tr})$ and $(Mat_n(\mathbb{C}), n^{-1}\mathrm{Tr})$ is identical, we make no distinction between their free cumulants. Lets denote by $NCP(p)$ the set of all non-crossing partitions over p letters which are also pair partitions. Recall that $NC(p)$ is a poset, the notion of partition refinement induces a partial order on $NC(p)$, which will be denoted by $\pi \le \sigma$ if, with $\pi, \sigma \in NC(p)$, each block of π is contained within a block of σ. Now a notion of the complement of a partition will be developed.

Definition 3.6. With $\pi \in NC(p_1)$, define the *non-crossing complement* $\pi^c \in NC(p_2)$ to be the unique non-crossing partition on p_2 letters so that $\pi\cup\pi^c \in NC(p_1+p_2)$, and $\sigma \le \pi^c$ for all other $\sigma \in NC(p_2)$ satisfying $\pi\cup\sigma \in NC(p_1+p_2)$.

Since the limiting spectral distribution of M is Wigner semicircle law with respect to the functional $n^{-1}\mathbb{E}\mathrm{Tr}$, and almost surely the Wigner semicircle law with respect to the functional $n^{-1}\mathrm{Tr}$, we have that $\kappa_2(M) = 1$ and $\kappa_p(M) = 0$ for $p \neq 2$. It follows now that

$$\langle x^k,\ x^q\rangle_{lr} = 0, \quad \text{if} \quad k+q \quad \text{is odd}, \tag{3.134}$$

and also that almost surely

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left\{\left(P^{(l)}MP^{(l)}\right)^k\left(P^{(r)}MP^{(r)}\right)^q\right\} = 0, \quad \text{if} \quad k+q \quad \text{is odd}. \tag{3.135}$$

Supposing then that $k+q$ is even, and continuing the calculation,

$$\begin{aligned}
\langle x^k,\ x^q \rangle_{lr} &= \sum_{\pi_2 \in NCP(even)} \sum_{\substack{\pi_1 \in NC(odd) \\ \pi_1 \cup \pi_2 \in NC(2(k+q)+1)}} \kappa_{\pi_1}(P^{(l)}, \cdots, P^{(r)}) \\
&= \sum_{\pi_2 \in NCP(k+q)} \sum_{\substack{\pi_1 \in NC(k+q+1) \\ \pi_1 \leq \pi_1^{\mathrm{c}}}} \kappa_{\pi_1}(P^{(l)}, \cdots, P^{(r)}) \\
&= \sum_{\pi_2 \in NCP(k+q)} \prod_{i=1}^{|\pi_1^{\mathrm{c}}|} \lim_{n\to\infty} \frac{1}{n} \mathbb{E}\mathrm{Tr}\{ \prod_{P^{(j)} \in S_i} P^{(j)} \},
\end{aligned} \tag{3.136}$$

where $\pi_1^{\mathrm{c}} = \{S_1, \cdots, S_{|\pi_1^{\mathrm{c}}|}\}$ are the blocks of the non-crossing complement of a given partition. We have used the complement partitions to write the sum of the free cumulants over the partitions of the projection matrices into a product of joint moments of the projection matrices.

Similarly, with respect to the functional $n^{-1}\mathrm{Tr}$, we have that almost surely

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \mathrm{Tr} \left\{ \left(P^{(l)} M P^{(l)} \right)^k \left(P^{(r)} M P^{(r)} \right)^q \right\} \\
&= \sum_{\pi_2 \in NCP(even)} \sum_{\substack{\pi_1 \in NC(odd) \\ \pi_1 \cup \pi_2 \in NC(2(k+q)+1)}} \kappa_{\pi_1}(P^{(l)}, \cdots, P^{(r)}) \\
&= \sum_{\pi_2 \in NCP(k+q)} \sum_{\substack{\pi_1 \in NC(k+q+1) \\ \pi_1 \leq \pi_1^{\mathrm{c}}}} \kappa_{\pi_1}(P^{(l)}, \cdots, P^{(r)}) \\
&= \sum_{\pi_2 \in NCP(k+q)} \prod_{i=1}^{|\pi_1^{\mathrm{c}}|} \lim_{n\to\infty} \frac{1}{n} \mathrm{Tr}\{ \prod_{P^{(j)} \in S_i} P^{(j)} \}.
\end{aligned} \tag{3.137}$$

Recall that the non-crossing pair partitions are in bijection with Dyck paths, $NCP(k+q) \to D_{(k+q)}$. Thus the computation for each functional reduces to counting Dyck paths. The number of Dyck paths $(h(0), \cdots, h(k+q))$ with $h(k) = j$ is

$$\left[\binom{k}{\frac{k+j}{2}} - \binom{k}{\frac{k+j+2}{2}} \right] \left[\binom{q}{\frac{q+j}{2}} - \binom{q}{\frac{q+j+2}{2}} \right] = \frac{(j+1)^2}{(k+1)(q+1)} \binom{k+1}{\frac{k+j+2}{2}} \binom{q+1}{\frac{q+j+2}{2}}.$$

Note that $\lim_{n\to\infty} n^{-1}\mathrm{Tr} \left(P^{(l)}\right)^a \left(P^{(r)}\right)^b = \gamma_{lr}$, for any $a, b \geq 1$. Also note that below the partition π_1^c depends on the Dyck path $d \in D_{(k+q)}$(which corresponds to some non-crossing pair partition). Also note that by $|\pi_1^c|$ we denote the number of blocks of π_1^c. Suppose for now that both k, q are even integers.

The height of the path at $h(k)$ must be even, say $h(k) = 2j$. Those blocks which consist only of the matrices $P^{(l)}$ will contribute a factor of γ_l to the product of joint moments. The number of

blocks which contain only the matrices $P^{(l)}$ corresponds to the number of down edges of the path in the first k steps. Denote by u the number of up edges and d the number of down edges of the path up to step k. Then $u+d=k$ and $u-d=2j$, which implies that $d=k/2-j$. The number of blocks which contain only the matrices $P^{(r)}$ is equal to the number of up edges of the path in the final q steps. This number corresponds to the exponent on the factor γ_r in the product of joint moments. Denote now by u the number of up edges and d the number of down edges of the path in the final q steps. The $u+d=q$ and $d-u=2j$, which implies that $u=q/2-j$. The remaining blocks of the partition contain projection matrices of mixed type and will contribute a factor γ_{lr} to the product of joint moments. Since the total number of blocks in the partition is $\frac{k+q}{2}+1$, the number of factors of γ_{lr} in the product of joint moments is $2j+1$. Partitioning the Dyck paths into equivalence classes based on the height $h(k)$, we get that

$$\begin{aligned}
\langle x^k, x^q\rangle_{lr} &= \sum_{d\in D_{(k+q)}} \prod_{i=1}^{|\pi_1^c|} \lim_{n\to\infty} \frac{1}{n}\mathbb{E}\mathrm{Tr}\{\prod_{P^{(j)}\in S_i} P^{(j)}\} \\
&= \sum_{j=0}^{\frac{k}{2}} \sum_{\substack{d\in D_{(k+q)}\\ h(k)=2j}} \gamma_l^{\frac{k}{2}-j}\gamma_r^{\frac{q}{2}-j}\gamma_{lr}^{2j+1} \\
&= \sum_{j=0}^{\frac{k}{2}} \frac{(2j+1)^2}{(k+1)(q+1)} \binom{k+1}{\frac{k+2j+2}{2}} \binom{q+1}{\frac{q+2j+2}{2}} \gamma_l^{\frac{k}{2}-j}\gamma_r^{\frac{q}{2}-j}\gamma_{lr}^{2j+1},
\end{aligned}$$

and also, almost surely,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left\{\left(P^{(l)}MP^{(l)}\right)^k\left(P^{(r)}MP^{(r)}\right)^q\right\} \\
&= \sum_{d\in D_{(k+q)}} \prod_{i=1}^{|\pi_1^c|} \lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}\{\prod_{P^{(j)}\in S_i} P^{(j)}\} \\
&= \sum_{j=0}^{\frac{k}{2}} \sum_{\substack{d\in D_{(k+q)}\\ h(k)=2j}} \gamma_l^{\frac{k}{2}-j}\gamma_r^{\frac{q}{2}-j}\gamma_{lr}^{2j+1} \\
&= \sum_{j=0}^{\frac{k}{2}} \frac{(2j+1)^2}{(k+1)(q+1)} \binom{k+1}{\frac{k+2j+2}{2}} \binom{q+1}{\frac{q+2j+2}{2}} \gamma_l^{\frac{k}{2}-j}\gamma_r^{\frac{q}{2}-j}\gamma_{lr}^{2j+1}.
\end{aligned}$$

Now suppose that both k,q are odd. The height of the path at $h(k)$ must be odd, say $h(k)=2j+1$. Similar to the even case, the number of blocks which consist only of the matrices $P^{(l)}$ equals the exponent of γ_l in the product of joint moments. The number of blocks which contain only the matrices $P^{(l)}$ corresponds to the number of down edges of the path in the first k steps. Denote by u the number of up edges and d the number of down edges of the path up to step k. Then $u+d=k$ and $u-d=2j+1$, which implies that $d=(k-1)/2-j$. The number of blocks which contain only the matrices $P^{(r)}$ is equal to the number of up edges of the path in the final q

steps. This number corresponds to the exponent on the factor γ_r in the product of joint moments. Denote now by u the number of up edges and d the number of down edges of the path in the final q steps. The $u+d=q$ and $d-u=2j+1$, which implies that $u=(q-1)/2-j$. The remaining blocks of the partition contain projection matrices of mixed type and will contribute a factor of γ_{lr} to the product of joint moments. Since the total number of blocks in the partition is $\frac{k+q}{2}+1$, the number of factors of γ_{lr} in the product of joint moments is $2j+2$. Partitioning the Dyck paths into equivalence classes based on the height $h(k)$, we get that

$$\begin{aligned}
\langle x^k, x^q\rangle_{lr} &= \sum_{d\in D_{(k+q)}} \prod_{i=1}^{|\pi_1^c|} \lim_{n\to\infty} \mathbb{E}\frac{1}{n}\mathrm{Tr}\{\prod_{P^{(j)}\in S_i} P^{(j)}\} \\
&= \sum_{j=0}^{\frac{k-1}{2}} \sum_{\substack{d\in D_{(k+q)} \\ h(k)=2j+1}} \gamma_l^{\frac{k-1}{2}-j}\gamma_r^{\frac{q-1}{2}-j}\gamma_{lr}^{2j+2} \\
&= \sum_{j=0}^{\frac{k-1}{2}} \frac{(2j+2)^2}{(k+1)(q+1)}\binom{k+1}{\frac{k+2j+3}{2}}\binom{q+1}{\frac{q+2j+3}{2}}\gamma_l^{\frac{k-1}{2}-j}\gamma_r^{\frac{q-1}{2}-j}\gamma_{lr}^{2j+2},
\end{aligned}$$

and also, almost surely,

$$\begin{aligned}
&\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left\{\left(P^{(l)}MP^{(l)}\right)^k\left(P^{(r)}MP^{(r)}\right)^q\right\} \\
&= \sum_{d\in D_{(k+q)}} \prod_{i=1}^{|\pi_1^c|} \lim_{n\to\infty} \frac{1}{n}\mathrm{Tr}\{\prod_{P^{(j)}\in S_i} P^{(j)}\} \\
&= \sum_{j=0}^{\frac{k-1}{2}} \sum_{\substack{d\in D_{(k+q)} \\ h(k)=2j+1}} \gamma_l^{\frac{k-1}{2}-j}\gamma_r^{\frac{q-1}{2}-j}\gamma_{lr}^{2j+2} \\
&= \sum_{j=0}^{\frac{k-1}{2}} \frac{(2j+2)^2}{(k+1)(q+1)}\binom{k+1}{\frac{k+2j+3}{2}}\binom{q+1}{\frac{q+2j+3}{2}}\gamma_l^{\frac{k-1}{2}-j}\gamma_r^{\frac{q-1}{2}-j}\gamma_{lr}^{2j+2}.
\end{aligned}$$

Now for polynomials $f(x)=\sum_{i=0}^p a_i x^i$ and $g(x)=\sum_{j=0}^m b_j x^j$, we have by linearity that

$$\langle f, g\rangle_{lr} = \sum_{i=0}^{p}\sum_{j=0}^{m} a_i b_j \langle x^i, x^j\rangle_{lr}. \tag{3.138}$$

The intersection of countably many events, each with probability 1, occurs with probability 1. There are only countably many polynomials with rational coefficients, so we have proved that the random variables

$$\frac{1}{n}\mathrm{Tr}\{P^{(l)}f(M^{(l)})P^{(l,r)}g(M^{(r)})P^{(r)}\},$$

converge almost surely to the same, non-random limit given by the right hand side of (3.138), whenever f,g are polynomials with rational coefficients.

In the next proposition the bilinear form $\langle f, g\rangle_{lr}$ is diagonalized.

Proposition 3.7. *The two families $\{U_k^{\gamma_l}\}_{k=0}^\infty$ and $\{U_q^{\gamma_r}\}_{q=0}^\infty$ of rescaled Chebyshev polynomials of the second kind are biorthogonal with respect to the bilinear form (3.131). More precisely,*

$$\frac{1}{\sqrt{\gamma_l\gamma_r}}\ \langle U_k^{\gamma_l}, U_q^{\gamma_r}\rangle_{lr} \ = \ \delta_{kq}\ (\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}})^{k+1}. \tag{3.139}$$

Proof. Since $\langle x^k,\ x^q\rangle_{lr} = 0$ if $k+q$ is odd, it follows by linearity that

$$\langle U_k^{\gamma_l}, U_q^{\gamma_r}\rangle_{lr} = 0, \quad \text{if } k+q \text{ is odd.} \tag{3.140}$$

We begin by computing $\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r}\rangle_{lr}$ and $\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r}\rangle_{lr}$. We obtain

$$\begin{aligned}
&\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r}\rangle_{lr} \\
&\quad = \ (\frac{1}{\sqrt{\gamma_l}})^{2k}\langle x^{2k}, U_{2q}^{\gamma_r}(x)\rangle_{lr} \\
&\quad = \ \gamma_l^{-k}\sum_{p=0}^{q}(-1)^p(\frac{1}{\sqrt{\gamma_l}})^{2q-2p}\binom{2q-p}{p}\langle x^{2k}, x^{2q-2p}\rangle_{lr} \\
&\quad = \ \frac{\gamma_l^{-k}\gamma_r^{-q}}{2k+1}\sum_{j=0}^{k}\sum_{p=0}^{q-j}\frac{(-1)^p\gamma_l^p(2j+1)^2}{2q-2p+1}\binom{2k+1}{k+j+1}\binom{2q-p}{p}\binom{2q-2p+1}{q-p+j+1}\gamma_l^{k-j}\gamma_r^{q-p-j}\gamma_{lr}^{2j+1} \\
&\quad = \ \frac{1}{2k+1}\sum_{j=0}^{k}(2j+1)^2\binom{2k+1}{k+j+1}\left[\sum_{p=0}^{q-j}\frac{(-1)^p(2q-p)!}{p!(q-p+j+1)!(q-p-j)!}\right]\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+1}
\end{aligned} \tag{3.141}$$

and

$$\begin{aligned}
&\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r}\rangle_{lr} \\
&\quad = \ (\frac{1}{\sqrt{\gamma_l}})^{2k+1}\langle x^{2k+1}, U_{2q+1}^{\gamma_r}(x)\rangle \\
&\quad = \ (\frac{1}{\sqrt{\gamma_l}})^{2k+1}\sum_{p=0}^{q}(-1)^p(\frac{1}{\sqrt{\gamma_r}})^{2q-2p+1}\binom{2q-p+1}{p}\langle x^{2k+1}, x^{2q-2p+1}\rangle_{lr} \\
&\quad = \ \frac{\gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}}{2k+2}\sum_{j=0}^{k}\sum_{p=0}^{q-j}\frac{(-1)^p\gamma_r^p(2j+2)^2}{2q-2p+2}\binom{2k+2}{k+j+2}\binom{2q-p+1}{p}\binom{2q-2p+2}{q-p+j+2}\gamma_l^{-j}\gamma_r^{-p-j}\gamma_{lr}^{2j+2} \\
&\quad = \ \frac{\gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}}{2k+2}\sum_{j=0}^{k}(2j+2)^2\binom{2k+2}{k+j+2}\left[\sum_{p=0}^{q-j}\frac{(-1)^p(2q-p+1)!}{p!(q-p+j+2)!(q-p-j)!}\right]\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+2}.
\end{aligned} \tag{3.142}$$

Denote by

$$H_1(q,j) = \sum_{p=0}^{q-j} \frac{(-1)^p (2q-p)!}{p!(q-p+j+1)!(q-p-j)!}, \tag{3.143}$$

$$H_2(q,j) = \sum_{p=0}^{q-j} \frac{(-1)^p (2q-p+1)!}{p!(q-p+j+2)!(q-p-j)!}. \tag{3.144}$$

Then

$$\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r} \rangle_{lr} = \frac{1}{2k+1} \sum_{j=0}^{k} (2j+1)^2 \binom{2k+1}{k+j+1} H_1(q,j) \gamma_l^{-j} \gamma_r^{-j} \gamma_{lr}^{2j+1}, \tag{3.145}$$

$$\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r} \rangle_{lr} = \frac{\gamma_l^{-\frac{1}{2}} \gamma_r^{-\frac{1}{2}}}{2k+2} \sum_{j=0}^{k} (2j+2)^2 \binom{2k+2}{k+j+2} H_2(q,j) \gamma_l^{-j} \gamma_r^{-j} \gamma_{lr}^{2j+2}. \tag{3.146}$$

It follows from (3.143-3.144) that

$$H_1(q,j) = \frac{(2q)!}{(q-j)!(q+j+1)!} = {}_2F_1 \left(\begin{matrix} -(q-j), -(q+j+1) \\ -2q \end{matrix} ; 1 \right), \tag{3.147}$$

$$H_2(q,j) = \frac{(2q+1)!}{(q-j)!(q+j+1)!} = {}_2F_1 \left(\begin{matrix} -(q-j), -(q+j+2) \\ -(2q+1) \end{matrix} ; 1 \right), \tag{3.148}$$

where ${}_2F_1$ is a hypergeometric function. See [3] for the definition of hypergeometric functions. Below let $(x)_n = x(x+1)\cdots(x+n-1)$ denote the rising factorial. By the Chu-Vandermonde identity (see e.g. [3]), it follows that

$$H_1(q,j) = \frac{(2q)!}{(q-j)!(q+j+1)!} \frac{(j-q+1)_{q-j}}{(-2q)_{q-j}} = \begin{cases} 0 & 0 \le j < q \\ \frac{1}{2q+1} & j = q \end{cases} \tag{3.149}$$

$$H_2(q,j) = \frac{(2q+1)!}{(q-j)!(q+j+2)!} \frac{(j-q+1)_{q-j}}{(-2q-1)_{q-j}} = \begin{cases} 0 & 0 \le j < q \\ \frac{1}{2q+2} & j = q \end{cases} \tag{3.150}$$

Therefore, for $k = 0, 1, \cdots, q-1$, we get that $\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2q}^{\gamma_r} \rangle_{lr} = 0$ and also $\langle (\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2q+1}^{\gamma_r} \rangle_{lr} = 0$. With $k = q$ we obtain

$$\begin{aligned} \langle (\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2k}^{\gamma_r} \rangle_{lr} &= \frac{1}{2k+1} \sum_{j=0}^{k} (2j+1)^2 \binom{2k+1}{k+j+1} H_1(k,j) \gamma_l^{-j} \gamma_r^{-j} \gamma_{lr}^{2j+1} \\ &= \frac{(2k+1)^2}{2k+1} \binom{2k+1}{2k+1} H_1(k,k) \gamma_l^{-k} \gamma_r^{-k} \gamma_{lr}^{2k+1} \\ &= \frac{\gamma_{lr}^{2k+1}}{\gamma_l^k \gamma_r^k} \end{aligned} \tag{3.151}$$

and

$$\begin{aligned}
\langle(\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2k+1}^{\gamma_r}\rangle_{lr} &= \frac{\gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}}{2k+2}\sum_{j=0}^{k}(2j+2)^2\binom{2k+2}{k+j+2}H_2(k,j)\gamma_l^{-j}\gamma_r^{-j}\gamma_{lr}^{2j+2} \\
&= \gamma_l^{-\frac{1}{2}}\gamma_r^{-\frac{1}{2}}\frac{(2k+1)^2}{2k+1}\binom{2k+1}{2k+1}H_2(k,k)\gamma_l^{-k}\gamma_r^{-k}\gamma_{lr}^{2k+1} \\
&= \frac{\gamma_{lr}^{2k+2}}{\sqrt{\gamma_l\gamma_r}^{2k+1}}. \qquad (3.152)
\end{aligned}$$

Thus, for $k < q$,

$$\langle U_{2k}^{\gamma_l}, U_{2q}^{\gamma_r}\rangle_{lr} = 0, \quad \langle U_{2k+1}^{\gamma_l}, U_{2q+1}^{\gamma_r}\rangle_{lr} = 0, \qquad (3.153)$$

and for $k = q$

$$\langle U_{2k}^{\gamma_l}, U_{2k}^{\gamma_r}\rangle_{lr} = \langle(\frac{x}{2\sqrt{\gamma_l}})^{2k}, U_{2k}^{\gamma_r}\rangle_{lr} = \frac{\gamma_{lr}^{2k+1}}{\gamma_l^k\gamma_r^k}, \qquad (3.154)$$

$$\langle U_{2k+1}^{\gamma_l}, U_{2k+1}^{\gamma_r}\rangle_{lr} = \langle(\frac{x}{2\sqrt{\gamma_l}})^{2k+1}, U_{2k+1}^{\gamma_r}\rangle_{lr} = \frac{\gamma_{lr}^{2k+2}}{\sqrt{\gamma_l\gamma_r}^{2k+1}}. \qquad (3.155)$$

This completes the proof of proposition 3.7, which is the diagonalization part of lemma 2.5.

Remark 3.8. Previously we have shown that whenever f, g are polynomials with rational coefficients, almost surely (a.s.)

$$\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\left\{P^{(l)}f(M^{(l)})\cdot P^{(l,r)}\cdot g(M^{(r)})P^{(r)}\right\} = \langle f, g\rangle_{lr}.$$

The Chebyshev polynomials have rational coefficients, so it follows from the above argument that a.s.

$$\frac{1}{\sqrt{\gamma_l\gamma_r}}\lim_{n\to\infty}\frac{1}{n}\mathrm{Tr}\{P^{(l)}U_k^{\gamma_l}(M^{(l)})P^{(l,r)}U_q^{\gamma_r}(M^{(r)})P^{(r)}\} = \delta_{kq}\ (\frac{\gamma_{lr}}{\sqrt{\gamma_l\gamma_r}})^{k+1}. \qquad (3.156)$$

□

Now the bilinear form $\langle\cdot,\cdot\rangle_{lr}$ will be extended to functions other than polynomials. For this part of the argument, the bound on the variance of linear eigenvalue statistics in 3.3 is essential.

Proposition 3.9. *Let $f, g \in \mathcal{H}_s$ for some $s > \frac{3}{2}$, i.e. for some $\epsilon > 0$,*

$$\int_{-\infty}^{\infty}|\widehat{f}(t)|^2\,(1+|t|)^{3+\epsilon}\,dt < \infty, \quad \int_{-\infty}^{\infty}|\widehat{g}(t)|^2\,(1+|t|)^{3+\epsilon}\,dt < \infty. \qquad (3.157)$$

Then the limit of $\langle f, g\rangle_{lr,n}$ (see definition 3.5) as $n \to \infty$ exists and

$$\langle f, g\rangle_{lr} = \frac{1}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} f(x)g(y)F_{lr}(x,y)\sqrt{4\gamma_l - x^2}\sqrt{4\gamma_r - y^2}dydx, \qquad (3.158)$$

and also, almost surely,

$$\lim_{n\to\infty}\frac{1}{n}Tr\left\{P^{(l)}f(M^{(l)})\cdot P^{(l,r)}\cdot g(M^{(r)})P^{(r)}\right\}$$
$$= \frac{1}{4\pi^2\gamma_l\gamma_r}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}}\int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} f(x)g(y)F_{lr}(x,y)\sqrt{4\gamma_l-x^2}\sqrt{4\gamma_r-y^2}dydx, \tag{3.159}$$

where the kernel $F_{lr}(x,y)$ is given by (3.88).

Proof. First it will be argued by approximation that $\langle\cdot,\cdot\rangle_{lr}$ can be extended to the class of functions $\mathcal{H}_{\frac{3}{2}+\epsilon}$, and then the bilinear form will be explicitly computed. It will be sufficient to approximate f,g below by truncated polynomials with rational coefficients in $\mathcal{H}_{\frac{3}{2}+\epsilon}$, because of the estimate 3.3 . Recall that functions of the Schwartz class are dense in $\mathcal{H}_s$, so after a triangle inequality argument it is in fact sufficient to suppose that $f,g\in\mathcal{S}(\mathbb{R})$. Let $h\in\mathcal{C}_c^\infty$ be a function so that $h(x)=1$ for $x\in[-3,3]$, $h(x)=0$ for $x\notin[-4,4]$ and is smoothly interpolated in between. Note that with overwhelming probability, the eigenvalues of the submatrices concentrate in the support of μ_{sc}. As a consequence we may suppose that f,g are supported in $[-3,3]$. We give a density argument. It is sufficient to argue that $||hf-hp_j||_{\frac{3}{2}+\epsilon}$ and $||hg-hq_j||_{\frac{3}{2}+\epsilon}$ converge to 0 as $j\to\infty$, where $\{p_j\},\{q_j\}$ are appropriately chosen sequences of polynomials with rational coefficients. Note that $hf=f$ and $hg=g$. We now focus on estimating $||f-hp_j||_{\frac{3}{2}+\epsilon}$. Since f is a Schwartz function, we have that $f\in\mathcal{H}_2$. We note that

$$\int_{-\infty}^{\infty}|\widehat{f}(t)|^2\,(1+|t|)^{3+\epsilon}\,dt\le\int_{-\infty}^{\infty}|\widehat{f}(t)|^2\,(1+|t|)^4\,dt, \tag{3.160}$$

so it will be sufficient to approximate f in the larger $||\cdot||_2$ norm. Also, since

$$||f||_2^2=\int_{-\infty}^{\infty}|\widehat{f}(t)|^2\,(1+|t|)^4\,dt\le Const\left[\int_{-\infty}^{\infty}|\widehat{f}(t)|^2dt+\int_{-\infty}^{\infty}t^4|\widehat{f}(t)|^2dt\right], \tag{3.161}$$

we only need to approximate the two terms on the right hand side. Consider polynomials $\{p_j\}$ with rational coefficients so that $\sup_{-4\le x\le 4}|f''(x)-p_j(x)|\to 0$ as $j\to\infty$. Then denote by $\tilde{p}_j(x)=\int_{-4}^x p_j(t)dt$, and $\tilde{\tilde{p}}_j(x)=\int_{-4}^x\tilde{p}_j(t)dt$. As a consequence of Parseval's theorem, it will be sufficient to show that

$$||f-h\tilde{\tilde{p}}_j||_{L^2([-4,4])}\to 0 \text{ and } ||f''-(h\tilde{\tilde{p}}_j)''||_{L^2([-4,4])}\to 0, \text{ as } j\to\infty. \tag{3.162}$$

But observe that

$$||f''-(h\tilde{\tilde{p}}_j)''||_{L^2([-4,4])}\ \le\ ||f''-hp_j||_{L^2([-4,4])}+||h''\tilde{\tilde{p}}_j+2h'\tilde{p}_j||_{L^2([-4,4])}. \tag{3.163}$$

The first term on the right hand side converges to 0 because of the uniform approximation. Noting that $h'(x)=0$ and $h''(x)=0$ on $(-3,3)$, and also that $\tilde{p}_j$ and $\tilde{\tilde{p}}_j$ converge to 0 uniformly on $[-4,-3)\cup(3,4]$, it follows that the second term on the right hand side converges to 0 as well.

Finally we observe that

$$\begin{aligned}
||f - h\tilde{\tilde{p}}_j||^2_{L^2([-4,4])} &= \int_{-4}^{4} |f(x) - h(x)\tilde{\tilde{p}}_j(x)|^2 dx \\
&\leq \int_{-4}^{4} h^2(x) \left| \int_{-4}^{x} \int_{-4}^{t} [f''(u) - p_j(u)] du dt \right|^2 dx \\
&\leq Const \cdot \left(\sup_{-4 \leq u \leq 4} \left| f''(u) - p_j(u) \right| \right)^2 \qquad (3.164)
\end{aligned}$$

It follows that $||f - h\tilde{\tilde{p}}_j||^2_{L^2([-4,4])} \to 0$ because of the uniform approximation. This completes the approximation argument, so we now turn toward computing the bilinear form.

Setting

$$f_k = \frac{1}{2\pi\gamma_l} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} f(x) U_k^{\gamma_l}(x) \sqrt{4\gamma_l - x^2} dx, \quad g_k = \frac{1}{2\pi\gamma_r} \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} g(y) U_k^{\gamma_r}(y) \sqrt{4\gamma_r - y^2} dy, \qquad (3.165)$$

it follows that

$$\begin{aligned}
&\langle f, g \rangle_{lr} \\
&= \langle \sum_{k=0}^{\infty} f_k U_k^{\gamma_l}(x), \sum_{p=0}^{\infty} g_p U_p^{\gamma_r}(x) \rangle_{lr} \\
&= \sum_{k=0}^{\infty} \sum_{p=0}^{\infty} f_k g_p \langle U_k^{\gamma_l}, U_p^{\gamma_r} \rangle_{lr} \\
&= \sum_{k=0}^{\infty} f_k g_k \frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2} \gamma_r^{k/2}} \\
&= \frac{1}{4\pi^2 \gamma_l \gamma_r} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} f(x) g(y) \left[\sum_{k=0}^{\infty} U_k^{\gamma_l}(x) U_k^{\gamma_r}(y) \frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2} \gamma_r^{k/2}} \right] \sqrt{4\gamma_l - x^2} \sqrt{4\gamma_r - y^2} dy dx.
\end{aligned}$$

It also follows, using (3.156), that a.s.

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \mathrm{Tr} \left\{ P^{(l)} f(M^{(l)}) \cdot P^{(l,r)} \cdot g(M^{(r)}) P^{(r)} \right\} \\
&= \frac{1}{4\pi^2 \gamma_l \gamma_r} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \int_{-2\sqrt{\gamma_r}}^{2\sqrt{\gamma_r}} f(x) g(y) \left[\sum_{k=0}^{\infty} U_k^{\gamma_l}(x) U_k^{\gamma_r}(y) \frac{\gamma_{lr}^{k+1}}{\gamma_l^{k/2} \gamma_r^{k/2}} \right] \sqrt{4\gamma_l - x^2} \sqrt{4\gamma_r - y^2} dy dx.
\end{aligned} \qquad (3.166)$$

Proposition 3.9 follows. □

Lemma 2.5 follows now from propositions 3.7 and 3.9. This also completes the proof of theorem 2.1.

4 Proof of Theorem 2.2

It is enough to show the case of $d = 2$, i.e. the limiting covariance of $\mathcal{N}_n^{(1)\circ}[\varphi_1]$ and $\mathcal{N}_n^{(2)\circ}[\varphi_2]$. Then by symmetry property, one will get the limiting covariance of $\mathcal{N}_n^{(l)\circ}[\varphi_1]$ and $\mathcal{N}_n^{(p)\circ}[\varphi_p]$, $1 \le l \le p \le d$. Let $U(t), \widetilde{U}(t), u_n(t), \widetilde{u}_n(t)$ be $U^{(1)}(t), U^{(2)}(t), u_n^{(1)}(t), u_n^{(2)}(t)$ defined in (3.16-3.17) respectively. $U(t)$ and $\widetilde{U}(t)$ are unitrary matrices and

$$U(t)U^*(t) = \widetilde{U}(t)\widetilde{U}^*(t) = I, \qquad |U_{jk}| \le 1, \qquad \sum_{k=1}^{n} |U_{jk}|^2 = 1.$$

By Remark 3.3 in [10], we have the following bounds

$$\mathbf{Var}\{u_n(t)\} \le C(\sigma_6)(1 + |t|^3)^2, \tag{4.1}$$

$$\mathbf{Var}\{\widetilde{u}_n(t)\} \le C(\sigma_6)(1 + |t|^3)^2, \tag{4.2}$$

$$\mathbf{Var}\{\mathcal{N}_n^{(1)\circ}(t)\} \le C(\sigma_6)\left(\int_{-\infty}^{\infty} (1 + |t|^3)|\widehat{\varphi}_1(t)|\, dt\right)^2, \tag{4.3}$$

$$\mathbf{Var}\{\mathcal{N}_n^{(2)\circ}(t)\} \le C(\sigma_6)\left(\int_{-\infty}^{\infty} (1 + |t|^3)|\widehat{\varphi}_2(t)|\, dt\right)^2. \tag{4.4}$$

Let w be a linear combination of random variables $\mathcal{N}_n^{(1)\circ}[\varphi_1]$ and $\mathcal{N}_n^{(2)\circ}[\varphi_2]$, and $Z_n(x)$ be the characteristic function of w, i.e.

$$w = \alpha\mathcal{N}_n^{(1)\circ}[\varphi_1] + \beta\mathcal{N}_n^{(2)\circ}[\varphi_2], \quad Z_n(x) = \mathbb{E}\{e^{ixw}\}. \tag{4.5}$$

So

$$Z_n(x) = 1 + \int_0^x Z_n'(t)dt;\ Z_n'(x) = i\mathbb{E}\{we^{ixw}\}, \tag{4.6}$$

By Cauchy-Schwarz inequality and (4.3-4.4)

$$|Z_n'(x)| \le (|\alpha| + |\beta|)C^{1/2}(\sigma_6)\int_{-\infty}^{\infty} (1 + |t|^3)|\widehat{\varphi}_1(t)|\, dt, \tag{4.7}$$

Fourier inversion formula $f(\lambda) = \int e^{it\lambda}\widehat{f}(t)\, dt$ givies us that

$$\mathcal{N}_n^{(1)\circ}[\varphi_1] = \int_{-\infty}^{\infty} \widehat{\varphi_1}(t)u_n^\circ(t)dt, \quad \mathcal{N}_n^{(2)\circ}[\varphi_2] = \int_{-\infty}^{\infty} \widehat{\varphi_2}(t)\widetilde{u}_n^\circ(t)dt. \tag{4.8}$$

So

$$w = \int_{-\infty}^{\infty} \alpha\widehat{\varphi_1}(t)u_n^\circ(t) + \beta\widehat{\varphi_2}(t)\widetilde{u}_n^\circ(t)dt, \tag{4.9}$$

$$Z_n'(x) = i\alpha\int_{-\infty}^{\infty} \widehat{\varphi_1}(t)Y_n(x,t)dt + i\beta\int_{-\infty}^{\infty} \widehat{\varphi_2}(t)\widetilde{Y}_n(x,t)dt, \tag{4.10}$$

where

$$Y_n(x,t) = \mathbb{E}\{u_n^\circ(t)e_n(x)\}, \quad \widetilde{Y}_n(x,t) = \mathbb{E}\{\widetilde{u}_n^\circ(t)e_n(x)\}, \quad e_n(x) = e^{ixw}. \tag{4.11}$$

By Cauchy-Schwarz inequality,

$$|Y_n(x,t)| \le \mathbb{E}\{|u_n^\circ(t)|\} \le C^{1/2}(\sigma_6)(1+|t|^3), \tag{4.12}$$
$$|\widetilde{Y}_n(x,t)| \le \mathbb{E}\{|\widetilde{u}_n^\circ(t)|\} \le C^{1/2}(\sigma_6)(1+|t|^3). \tag{4.13}$$

and

$$\begin{aligned} |\frac{\partial}{\partial x} Y_n(x,t)| &= |\mathbb{E}\{\alpha u_n^\circ \mathcal{N}_n^{(1)\circ}[\varphi_1] e_n(x) + \beta u_n^\circ \mathcal{N}_n^{(2)\circ}[\varphi_2] e_n(x)\}| \\ &\le C(\sigma_6)(1+|t|^3) \int_{-\infty}^{\infty} (1+|t|^3)(|\alpha \widehat{\varphi}_1(t)| + |\beta \widehat{\varphi}_2(t)|) dt. \end{aligned} \tag{4.14}$$

And

$$\frac{\partial}{\partial t} Y_n(x,t) = \mathbb{E}\{u_n'(t) e_n^\circ(x)\} = \frac{i}{\sqrt{n}} \sum_{j,k \in B_1} \mathbb{E}\{W_{jk} \Phi_n\}. \tag{4.15}$$

where

$$\Phi_n = U_{jk}(t) e_n^\circ(x).$$

Recall $D_{jk} = \partial/\partial M_{jk}$, $\beta_{jk} = (1+\delta_{jk})^{-1}$,

$$D_{jk} U_{ab}(t) = 1_{j,k\in B_1} i\beta_{jk} [U_{aj} * U_{bk}(t) + U_{bj} * U_{ak}(t)], \tag{4.16}$$
$$D_{jk} \widetilde{U}_{ab}(t) = 1_{j,k\in B_2} i\beta_{jk} [\widetilde{U}_{aj} * \widetilde{U}_{bk}(t) + \widetilde{U}_{bj} * \widetilde{U}_{ak}(t)]. \tag{4.17}$$

and

$$D_{jk} e_n(x) = 2i\beta_{jk} x e_n(x)(1_{j,k\in B_1} \alpha(\varphi_1)'_{jk}(M_1) + 1_{j,k\in B_2} \beta(\varphi_2)'_{jk}(M_2)) \tag{4.18}$$
$$= -2\beta_{jk} x e_n(x) \left(1_{j,k\in B_1} \int_{-\infty}^{\infty} t U_{jk}(t) \alpha \widehat{\varphi_1}(t) dt + 1_{j,k\in B_2} \int_{-\infty}^{\infty} t \widetilde{U}_{jk} \beta \widehat{\varphi_2}(t) dt \right) \tag{4.19}$$

Lemma 4.1. *Assume φ_1, φ_2 have fourth bounded derivatives. Then*

$$|D_{jk}^l (U_{jk}(t) e_n^\circ(x))| \le C_l(x,t), \quad 0 \le l \le 5 \tag{4.20}$$

where $C_l(x,t)$ is a degree l polynomial of $|x|, |t|$ with positive coefficients.

Proof. From (4.16) and (4.17), we have

$$|D_{jk}^l U_{ab}(t)|, |D_{jk}^l \widetilde{U}_{ab}(t)| \le Const_l |t|^l, \quad 0 \le l \le 5. \tag{4.21}$$

(4.19) implies

$$|D_{jk}^l e_n(x)| \le Const_l'(1+|x|^l) \quad 0 \le l \le 5. \tag{4.22}$$

These two complete the proof of Lemma 4.1 □

Apply (A.1) with $p=2$ to obtain

$$\begin{aligned} \frac{\partial}{\partial t} Y_n(x,t) &= \frac{i}{n} \sum_{j,k\in B_1} (1 + (\sigma^2 - 1)\delta_{jk}) \mathbb{E}\{D_{jk} \Phi_n\} + O(1) \\ &= \frac{i}{n} \sum_{j,k\in B_1} (1+\delta_{jk}) \mathbb{E}\{D_{jk}\Phi_n\} + \frac{i(\sigma^2-2)}{n} \sum_{j\in B_1} \mathbb{E}\{D_{jj}\Phi_n\} + O(1). \end{aligned} \tag{4.23}$$

where the error term is bounded by $C_3(x,t)$ as $n\to\infty$. The first term in (4.23) is

$$-\frac{t}{n}Y_n(t,x)-\frac{1}{n}\int_0^t\mathbb{E}\{u_n(t-t_1)\}Y_n(x,t_1)dt_1-\frac{1}{n}\mathbb{E}\{\int_0^t u_n(t_1)u_n^\circ(t-t_1)dt_1e_n^\circ(x)\}$$
$$-\frac{2i}{n}\mathbb{E}\{xe_n(x)\left(\int_{-\infty}^{\infty}t_1u_n(t+t_1)\alpha\widehat{\varphi_1}(t_1)dt_1+\int_{-\infty}^{\infty}t_1TrP^{(1,2)}U_n(t)P^{(1,2)}\widetilde{U}_n(t_1)\beta\widehat{\varphi_2}(t_1)dt_1\right).$$

The first term and the second term are bounded because of (4.12). The last term is bounded by $2|x|\int_{-\infty}^{\infty}|t|(|\alpha||\widehat{\varphi}_1(t_1)|+|\beta||\widehat{\varphi}_2(t_1)|)dt_1$. And the third term is bounded by $2|t|C^{1/2}(\sigma_6)(1+|t|^3)$.

The second term in (4.23) is

$$\frac{2-\sigma^2}{n}\sum_{j\in B_1}\mathbb{E}\left\{\int_0^t U_{jj}(t_1)U_{jj}(t-t_1)dt_1e_n^\circ(x)\right\}+\frac{ix(2-\sigma^2)}{n}\sum_{j\in B_1}\mathbb{E}\left\{e_n(x)\int_{-\infty}^{\infty}t_1U_{jj}(t)U_{jj}(t_1)\alpha\widehat{\varphi_1}(t_1)dt_1\right\}$$
$$+\frac{ix(2-\sigma^2)}{n}\sum_{j\in B_1\cap B_2}\mathbb{E}\left\{e_n(x)\int_{-\infty}^{\infty}t_1U_{jj}(t)\widetilde{U}_{jj}(t_1)\beta\widehat{\varphi_2}(t_1)dt_1\right\}$$

The first term is bounded by $2|2-\sigma^2||t|$, and the second term is bounded by $2|x||2-\sigma^2|\int_{-\infty}^{\infty}|t|(|\alpha||\widehat{\varphi}_1(t_1)|+|\beta||\widehat{\varphi}_2(t_1)|)dt_1$. So

$$\left|\frac{\partial}{\partial t}Y_n(x,t)\right|\le C_5(x,t).$$

Symmetricly, $\widetilde{Y}_n(x,t)$ has similar bounds. So we conclude that the sequences $\{Y_n\},\{\widetilde{Y}_n\}$ are bounded and equicontinuous on any finite subset of $\mathbb{R}^2$. We will prove now that any uniformly converging subsequence of $\{Y_n\}(\{\widetilde{Y}_n\})$ has same limit $Y(\widetilde{Y})$.

We deal with Y_n first, and by the symmetric property, we can find $\widetilde{Y}_n$. By identity,

$$u_n(t)=n_1+i\int_0^t\sum_{j,k\in B_1}M_{jk}U_{jk}(t_1)dt_1, \tag{4.24}$$

so

$$Y_n(x,t)=\frac{i}{\sqrt{n}}\int_0^t\sum_{j,k\in B_1}\mathbb{E}\{W_{jk}U_{jk}(t_1)e_n^\circ(x)\}dt_1. \tag{4.25}$$

By applying decoupling formula (A.1) with $p=3$ to (4.25), we have

$$Y_n(x,t)=\frac{i}{\sqrt{n}}\int_0^t\sum_{j,k\in B_1}\left[\sum_{l=0}^{3}\frac{\kappa_{l+1,jk}}{n^{l/2}l!}\mathbb{E}\{D_{jk}^l(U_{jk}(t_1)e_n^\circ(x))\}+\varepsilon_{3,jk}\right]dt_1, \tag{4.26}$$

where

$$\kappa_{1,jk}=0,\kappa_{2,jk}=1+\delta_{jk}(\sigma^2-1), \tag{4.27}$$
$$\kappa_{3,jk}=\mu_3,\kappa_{4,jk}=\kappa_4,j\ne k, \tag{4.28}$$

and $\kappa_{3,jj},\kappa_{4,jj}$ are uniformly bounded, i.e. there exist constants σ_3,σ_4 such that

$$|\kappa_{3,jj}|\le\sigma_3,|\kappa_{4,jj}|\le\sigma_4, \tag{4.29}$$

and

$$|\varepsilon_{3,jk}|\le n^{-2}C_3\mathbb{E}\{|W_{jk}|^5\}\sup_{t\in\mathbb{R}}\left|D_{jk}^4\Phi_n(x)\right|\le n^{-2}C_4(x,t). \tag{4.30}$$

Let

$$T_l = \frac{i}{n^{(l+1)/2}} \int_0^t \sum_{j,k \in B_1} \frac{\kappa_{l+1,jk}}{l!} \mathbb{E}\{D_{jk}^l(U_{jk}(t_1)e_n^\circ(x))\}dt_1, l = 1,2,3, \tag{4.31}$$

$$\mathcal{E}_n = \frac{i}{\sqrt{n}} \int_0^t \sum_{j,k \in B_1} \varepsilon_{3,jk} dt. \tag{4.32}$$

Then

$$Y_n(x,t) = T_1 + T_2 + T_3 + \mathcal{E}_n,$$

and

$$|\mathcal{E}_n| \le \frac{n_1^2}{n^{5/2}} C_5(x,t) \to 0, \quad as\ n \to \infty.$$

Note: if W_{jk}'s are Gaussian, then $Y_n(x,t) = T_1$. So T_1 coincide with the Y_n in Theorem 2.1.

Let

$$\bar{v}_n(t) = n^{-1}\mathbb{E}\{u_n(t)\}, \quad \bar{\widetilde{v}}_n(t) = n^{-1}\mathbb{E}\{\widetilde{u}_n(t)\}.$$

Then

$$Y_n(x,t) + 2\int_0^t dt_1 \int_0^{t_1} \bar{v}_n(t_1 - t_2) Y_n(x,t_2)dt_2 = xZ_n(x)A_n(t) + r_n(x,t) + T_2 + T_3 + \mathcal{E}_n, \tag{4.33}$$

where

$$A_n(t) = -\frac{2\alpha}{n}\int_0^t \mathbb{E}\{TrU(t_1)P_1\varphi_1'(M_1)P_1\}dt_1 - \frac{2\beta}{n}\int_0^t \mathbb{E}\{TrU(t_1)P_2\varphi_2'(M_2)P_2\}dt_1, \tag{4.34}$$

and $r_n(x,t) \to 0$ on any bounded subset of $\{(x,t) : x \in \mathbb{R}, t > 0\}$.

Let $A(t) = \lim_{n\to\infty} A_n(t)$. From the proof of Theorem 2.1, $A(t)$ coincides with the one in Gaussian case.

Proposition 4.2. $T_2 \to 0$ *on any bounded subset of* $\{(x,t) : x \in \mathbb{R}, t > 0\}$.

Proof. The second derivative (l=2) is

$$\begin{aligned}
&D_{jk}^2(U_{jk}(t_1)e_n^\circ(x)) = \beta_{jk}^2 \times \\
&\quad \{-(6U_{jj} * U_{jk} * U_{kk} + 2U_{jk} * U_{jk} * U_{jk})(t_1)e_n^\circ(x) \\
&\quad -4i(U_{jj} * U_{kk} + U_{jk} * U_{jk})(t_1)xe_n(x)\left[\int_{-\infty}^{\infty} tU_{jk}(t)\alpha\widehat{\varphi_1}(t)dt + 1_{j,k\in B_2}\int_{-\infty}^{\infty} t\widetilde{U}_{jk}\beta\widehat{\varphi_2}(t)dt\right] \\
&\quad +4U_{jk}(t_1)x^2 e_n(x)\left[\int_{-\infty}^{\infty} tU_{jk}(t)\alpha\widehat{\varphi_1}(t)dt + 1_{j,k\in B_2}\int_{-\infty}^{\infty} t\widetilde{U}_{jk}\beta\widehat{\varphi_2}(t)dt\right]^2 \\
&\quad -2iU_{jk}(t_1)xe_n(x)\left[\int_{-\infty}^{\infty} t(U_{jj} * U_{kk} + U_{jk} * U_{jk})(t)\alpha\widehat{\varphi_1}(t)dt\right. \\
&\quad \left. +1_{j,k\in B_2}\int_{-\infty}^{\infty} t(\widetilde{U}_{jj} * \widetilde{U}_{kk} + \widetilde{U}_{jk} * \widetilde{U}_{jk})(t)\beta\widehat{\varphi_2}(t)dt\right]\}.
\end{aligned}$$

Let

$$
\begin{aligned}
T_{21} \quad = \quad & \frac{i\kappa_3}{2n^{3/2}} \int_0^t \mathbb{E}\Bigg\{ \sum_{j,k\in B_1} -\beta_{jk}^2 (6U_{jj} * U_{jk} * U_{kk} + 2U_{jk} * U_{jk} * U_{jk})(t_1) e_n^\circ(x) \\
& -4i\beta_{jk}^2 (U_{jj} * U_{kk} + U_{jk} * U_{jk})(t_1) x e_n(x) \int t_2 U_{jk}(t_2) \alpha \widehat{\varphi_1}(t_2) dt_2 \\
& +4\beta_{jk}^2 U_{jk}(t_1) x^2 e_n(x) (\int t_2 U_{jk}(t_2) \alpha \widehat{\varphi_1}(t_2) dt_2)^2 \\
& -2i\beta_{jk}^2 U_{jk}(t_1) x e_n(x) \int t_2 (U_{jj} * U_{kk} + U_{jk} * U_{jk})(t_2) \beta \widehat{\varphi_1}(t_2) dt_2 \Bigg\} dt_1,
\end{aligned}
$$

$$
\begin{aligned}
T_{22} \quad = \quad & \frac{i\kappa_3}{2n^{3/2}} \int_0^t \mathbb{E}\Bigg\{ \sum_{j,k\in B_1\cap B_2} 4\beta_{jk}^2 U_{jk}(t_1) x^2 e_n(x) (\int t_2 \widetilde{U}_{jk}(t_2) \beta \widehat{\varphi_2}(t_2) dt_2)^2 \\
& +8\beta_{jk}^2 U_{jk}(t_1) x^2 e_n(x) \int t_2 U_{jk}(t_2) \alpha \widehat{\varphi_1}(t_2) dt_2 \int t_3 \widetilde{U}_{jk}(t_3) \beta \widehat{\varphi_2}(t_3) dt_3 \\
& -2i\beta_{jk}^2 U_{jk}(t_1) x e_n(x) \int t_2 [\widetilde{U}_{jj} * \widetilde{U}_{kk} + \widetilde{U}_{jk} * \widetilde{U}_{jk}](t_2) \beta \widehat{\varphi_2}(t_2) dt_2 \Bigg\} dt_1,
\end{aligned}
$$

$$
T_{23} \quad = \quad \frac{i}{2n^{3/2}} \int_0^t \sum_{j\in B_1} \kappa_{3,jj} \mathbb{E}\{D_{jj}^2 (U_{jj}(t_1) e_n^\circ(x))\} dt_1.
$$

Then $T_2 = T_{21} + T_{22} + T_{23}$. In [10], it has already shown that $|T_{21}| \le |t| C_2(x,t) n_1/n^{3/2}$ on any bounded subset of $\{(x,t) : x \in \mathbb{R}, t > 0\}$. Also, by Propersition 4.1 and (4.29), one has $|T_{23}| \le |t| C_2(x,t) n_1/n^{3/2}$.

In T_{22}, there are three types of sum,

$$
\begin{aligned}
S_1 &= n^{-3/2} \sum_{j,k\in B_1\cap B_2} U_{jk}(t_1) \widetilde{U}_{jk}(t_2) \widetilde{U}_{jk}(t_3), \\
S_2 &= n^{-3/2} \sum_{j,k\in B_1\cap B_2} U_{jk}(t_1) U_{jk}(t_2) \widetilde{U}_{jk}(t_3), \\
S_3 &= n^{-3/2} \sum_{j,k\in B_1\cap B_2} U_{jk}(t_1) \widetilde{U}_{jj}(t_2) \widetilde{U}_{kk}(t_3).
\end{aligned}
$$

By the Schwarz inequality that

$$
\begin{aligned}
|S_1| &\le n^{-3/2} \sum_{j,k\in B_2} |\widetilde{U}_{jk}(t_2) \widetilde{U}_{jk}(t_3)| \le \frac{n_2}{n^{3/2}}, \\
|S_2| &\le n^{-3/2} \sum_{j,k\in B_1} |U_{jk}(t_1) U_{jk}(t_2)| \le \frac{n_1}{n^{3/2}}.
\end{aligned}
$$

Write

$$
S_3 = \frac{n_{12}}{n^{3/2}} (P_{12} U(t_1) P_{12} V(t_2), V(t_3)),
$$

where

$$V(t) = n_{12}^{-1/2}(\widetilde{U}_{jj}(t))^T_{j\in B_1\cap B_2}.$$

$\|V(t)\| \leq 1$, $\|P_{12}U(t)P_{12}\| \leq 1$, so we conclude that $S_3 \leq \frac{n_{12}}{n^{3/2}}$, hence $T_{22} \leq |t|/n^{3/2}$. This completes the proof of preposition 4.2. $\square$

Proposition 4.3.

$$T_3 = T_{31} + T_{32} + R_3(x,t),$$

where

$$T_{31} = \frac{i\kappa_4}{n^2}\int_0^t \sum_{j,k\in B_1} \mathbb{E}\left\{U_{jj} * U_{kk}(t_1)xe_n(x)\int t_2U_{jj} * U_{kk}(t_2)\alpha\widehat{\varphi_1}(t_2)dt_2\right\}dt_1,$$

$$T_{32} = \frac{i\kappa_4}{n^2}\int_0^t \sum_{j,k\in B_1\cap B_2} \mathbb{E}\left\{U_{jj} * U_{kk}(t_1)xe_n(x)\int t_2\widetilde{U}_{jj} * \widetilde{U}_{kk}(t_2)\beta\widehat{\varphi_2}(t_2)dt_2\right\}dt_1.$$

and $R_3(x,t) \to 0$ *on any bounded subset of* $\{(x,t) : x\in\mathbb{R}, t>0\}$.

Proof.

$$T_3 = \frac{i\kappa_4}{6n^2}\int_0^t \sum_{j,k\in B_1}\mathbb{E}\{D^3_{jk}(U_{jk}(t_1)e^\circ_n(x))\}dt_1 + \widetilde{T}_3,$$

where

$$\widetilde{T}_3 = \frac{i}{6n^2}\int_0^t \sum_{j\in B_1}(\kappa_{4,jj}-\kappa_4)\mathbb{E}\{D^3_{jj}(U_{jj}(t_1)e^\circ_n(x))\}dt_1.$$

By Propersition 4.1 and (4.29), we have $|\widetilde{T}_3| \leq |t|C_3(x,t)n_1/n^2$.

The third derivative (l=3)

$$
\begin{aligned}
&D^3_{jk}(U_{jk}(t_1)e^\circ_n(x)) = \beta^3_{jk}\times \\
&\quad\{-i(36U_{jj}*U_{jk}*U_{jk}*U_{kk}+6U_{jj}*U_{jj}*U_{kk}*U_{kk}+6U_{jk}*U_{jk}*U_{jk}*U_{jk})(t_1)e^\circ_n(x) \\
&\quad+6(6U_{jj}*U_{jk}*U_{kk}+2U_{jk}U_{jk}*U_{jk})(t_1)xe_n(x)\left(\int tU_{jk}(t)\alpha\widehat{\varphi_1}(t)dt\right. \\
&\quad\left.+1_{j,k\in B_2}\int t\widetilde{U}_{jk}\beta\widehat{\varphi_2}(t)dt\right) \\
&\quad+12i(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t_1)x^2e_n(x)\left(\int tU_{jk}(t)\alpha\widehat{\varphi_1}(t)dt+1_{j,k\in B_2}\int t\widetilde{U}_{jk}\beta\widehat{\varphi_2}(t)dt\right)^2 \\
&\quad+6(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t_1)xe_n(x)\left(\int t(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t)\alpha\widehat{\varphi_1}(t)dt\right. \\
&\quad\left.+1_{j,k\in B_2}\int t(\widetilde{U}_{jj}*\widetilde{U}_{kk}+\widetilde{U}_{jk}*\widetilde{U}_{jk})\beta\widehat{\varphi_2}(t)dt\right) \\
&\quad-8U_{jk}(t_1)x^3e_n(x)\left(\int tU_{jk}(t)\alpha\widehat{\varphi_1}(t)dt+1_{j,k\in B_2}\int t\widetilde{U}_{jk}\beta\widehat{\varphi_2}(t)dt\right)^3 \\
&\quad+12iU_{jk}(t_1)x^2e_n(x)\left(\int tU_{jk}(t)\alpha\widehat{\varphi_1}(t)dt+1_{j,k\in B_2}\int t\widetilde{U}_{jk}\beta\widehat{\varphi_2}(t)dt\right) \\
&\quad\times\left(\int t(U_{jj}*U_{kk}+U_{jk}*U_{jk})(t)\alpha\widehat{\varphi_1}(t)dt+1_{j,k\in B_2}\int t(\widetilde{U}_{jj}*\widetilde{U}_{kk}+\widetilde{U}_{jk}*\widetilde{U}_{jk})\beta\widehat{\varphi_2}(t)dt\right) \\
&\quad+2U_{jk}(t_1)xe_n(x)\left[\int t(6U_{jj}*U_{jk}*U_{kk}+2U_{jk}*U_{jk}*U_{jk})(t)\alpha\widehat{\varphi_1}(t)dt\right. \\
&\quad\left.+1_{j,k\in B_2}\int t(6\widetilde{U}_{jj}*\widetilde{U}_{jk}*\widetilde{U}_{kk}+2\widetilde{U}_{jk}*\widetilde{U}_{jk}*\widetilde{U}_{jk})(t)\beta\widehat{\varphi_2}(t)dt\right]\}.
\end{aligned}
\tag{4.35}
$$

So any term of

$$\frac{i\kappa_4}{6n^2}\int_0^t\sum_{j,k\in B_1}\mathbb{E}\{D^3_{jk}(U_{jk}(t_1)e^\circ_n(x))\}dt_1$$

containing at least one off-diagonal entry U_{jk} or $\widetilde{U}_{jk}$ is bounded by $C_3(x,t)n_1/n^2$. Let $R_3(x,t)$ be the sum of $\widetilde{T}_3$ and these terms. Then $|R_3(x,t)|\le C_3(x,t)n_1/n^2+|t|C_3(x,t)n_1/n^2$. So two terms in (4.35) containg diagonal entries of U and $\widetilde{U}$ only left contribute to T_3. They are T_{31} and T_{32}. $\square$

Let

$$v(t)=\frac{1}{2\pi\gamma_1}\int_{-2\sqrt{\gamma_1}}^{2\sqrt{\gamma_1}}e^{it\lambda}\sqrt{4\gamma_1-\lambda^2}d\lambda,\quad \widetilde{v}(t)=\frac{1}{2\pi\gamma_2}\int_{-2\sqrt{\gamma_2}}^{2\sqrt{\gamma_2}}e^{it\lambda}\sqrt{4\gamma_2-\lambda^2}d\lambda.$$

By Wigner semicircle law, one has

$$\lim_{n\to\infty}\bar{v}_n(t)=\gamma_1v(t),\quad \lim_{n\to\infty}\bar{\widetilde{v}}_n(t)=\gamma_2\widetilde{v}(t).$$

Then

$$(v*v)(t)=-\frac{i}{2\pi\gamma_1^2}\int_{-2\sqrt{\gamma_1}}^{2\sqrt{\gamma_1}}e^{it\mu}\mu\sqrt{4\gamma_1-\mu^2}d\mu=\frac{1}{\pi t\gamma_1^2}\int_{-2\sqrt{\gamma_1}}^{2\sqrt{\gamma_1}}e^{it\mu}\frac{2\gamma_1-\mu^2}{\sqrt{4\gamma_1-\mu^2}}d\mu, \tag{4.36}$$

$$(\widetilde{v}*\widetilde{v})(t) = -\frac{i}{2\pi\gamma_2^2}\int_{-2\sqrt{\gamma_2}}^{2\sqrt{\gamma_2}} e^{it\mu}\mu\sqrt{4\gamma_2-\mu^2}d\mu = \frac{1}{\pi t\gamma_2^2}\int_{-2\sqrt{\gamma_2}}^{2\sqrt{\gamma_2}} e^{it\mu}\frac{2\gamma_2-\mu^2}{\sqrt{4\gamma_2-\mu^2}}d\mu. \tag{4.37}$$

Let

$$I(t) = \int_0^t (v*v)(t_1)dt_1, \quad \widetilde{I}(t) = \int_0^t (\widetilde{v}*\widetilde{v})(t_1)dt_1. \tag{4.38}$$

Denote

$$B_{\varphi_l} = \frac{1}{\pi\gamma_l^2}\int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \varphi_l(\mu)\frac{2\gamma_l-\mu^2}{\sqrt{4\gamma_l-\mu^2}}d\mu, l=1,2. \tag{4.39}$$

Proposition 4.4.

$$T_{31} \to i\kappa_4 xZ(x)I(t)\alpha\gamma_1^2 B_{\varphi_1}, \tag{4.40}$$

$$T_{32} \to i\kappa_4 xZ(x)I(t)\beta\gamma_{12}^2 B_{\varphi_2}. \tag{4.41}$$

uniformly on any bounded subset of $\{(x,t): x\in\mathbb{R}, t>0\}$.

Proof. The proof of (4.40) can be found in [10]. And

$$\begin{aligned} T_{32} &= \frac{ix\kappa_4}{n^2}\int_0^t \sum_{j,k\in B_1\cap B_2}\int_0^{t_1}\int\int_0^{t_2} t_2\mathbb{E}\{U_{jj}(t_3)U_{kk}(t_1-t_3)\widetilde{U}_{jj}(t_4)\widetilde{U}_{kk}(t_2-t_4)e_n(x)\} \\ &\quad\times\beta\widehat{\varphi_2}(t_2)dt_4dt_2dt_3dt_1 \\ &= ix\kappa_4\int_0^t\int_0^{t_1}\int\int_0^{t_2} t_2\mathbb{E}\{v_n(t_3,t_4)v_n(t_1-t_3,t_2-t_4)e_n^\circ(x)\}\beta\widehat{\varphi_2}(t_2)dt_4dt_2dt_3dt_1 \\ &\quad+ix\kappa_4 Z_n(x)\int_0^t\int_0^{t_1}\int\int_0^{t_2} t_2\mathbb{E}\{v_n(t_3,t_4)v_n(t_1-t_3,t_2-t_4)\}\beta\widehat{\varphi_2}(t_2)dt_4dt_2dt_3dt_1 \end{aligned} \tag{4.42}$$

where

$$v_n(t_1,t_2) = n^{-1}\sum_{j\in B_1\cap B_2} U_{jj}(t_1)\widetilde{U}_{jj}(t_2). \tag{4.43}$$

Then

$$|\mathbb{E}\{v_n(t_1,t_2)v_n(t_3,t_4)e_n^\circ(x)\}| \le 4\mathbb{E}\{|v_n^\circ(t_1,t_2)|\} + 4\mathbb{E}\{|v_n^\circ(t_3,t_4)|\}, \tag{4.44}$$

and

$$\mathbb{E}\{v_n(t_1,t_2)v_n(t_3,t_4)\} = \bar{v}_n(t_1,t_2)\bar{v}_n(t_3,t_4) + \mathbb{E}\{v_n(t_1,t_2)v_n^\circ(t_3,t_4)\}, \tag{4.45}$$

where

$$\bar{v}_n(t_1,t_2) = \mathbb{E}\{v_n(t_1,t_2)\}. \tag{4.46}$$

Proposition 4.5.

$$\bar{v}_n(t_1,t_2) = \gamma_{12}v(t_1)\widetilde{v}(t_2) + o(1),$$

uniformly on any compact set of $\mathbb{R}^2$.

Proof. Indeed, $\mathbb{E}\{U_{jj}(t_1)\widetilde{U}_{jj}(t_2)\} = v(t_1)\widetilde{v}(t_2) + o(1)$ uniformly in $1\le j\le n$ and t_1,t_2 from a compact set of $\mathbb{R}^2$, which follows from

$$\begin{aligned} \mathbb{E}U_{jj}(t) &= v(t)+o(1), \quad \mathbf{Var}\{U_{jj}(t)\} = o(1), \\ \mathbb{E}\widetilde{U}_{jj}(t) &= \widetilde{v}(t)+o(1), \quad \mathbf{Var}\{\widetilde{U}_{jj}(t)\} = o(1) \end{aligned}$$

(see e.g. [14]). □

So the limit of T_{32} is

$$ix\kappa_4 Z(x)\gamma_{12}^2 \int_0^t v * v(t_1)dt_1 \int_{-\infty}^{\infty} t_2\beta\widehat{\varphi_2}(t_2)\widetilde{v} * \widetilde{v}(t_2)dt_2 = ix\kappa_4 Z(x)\gamma_{12}^2 I(t)\beta B_{\varphi_2}.$$

$\square$

So if $Y(x,t) = \lim_{n\to\infty} Y_n(x,t)$, then $Y(x,t)$ satisfies

$$Y(x,t) + 2\gamma_1 \int_{-\infty}^{\infty} dt_1 \int_0^{t_1} v(t_1 - t_2)Y(x,t_2)dt_2 = xZ(x)\left[A(t) + i\kappa_4 I(t)(\alpha\gamma_1^2 B_{\varphi_1} + \beta\gamma_{12}^2 B_{\varphi_2})\right].$$

Therefore, if let $Y^*(x,t)$ be the solution of

$$Y(x,t) + 2\gamma_1 \int_{-\infty}^{\infty} dt_1 \int_0^{t_1} v(t_1 - t_2)Y(x,t_2)dt_2 = xZ(x)A(t),$$

then

$$Y(x,t) = Y^*(x,t) + \frac{i\kappa_4 xZ(x)}{2\pi\gamma_1^2}\left[\alpha\gamma_1^2 B_{\varphi_1} + \beta\gamma_{12}^2 B_{\varphi_2}\right] \int_{-2\sqrt{\gamma_1}}^{2\sqrt{\gamma_1}} \frac{e^{it\lambda}(2\gamma_1 - \lambda^2)}{\sqrt{4\gamma_1 - \lambda^2}} d\lambda. \tag{4.47}$$

Symmetrically,

$$\widetilde{Y}(x,t) = \widetilde{Y}^*(x,t) + \frac{i\kappa_4 xZ(x)}{2\pi\gamma_2^2}\left[\alpha\gamma_{12}^2 B_{\varphi_1} + \beta\gamma_2^2 B_{\varphi_2}\right] \int_{-2\sqrt{\gamma_2}}^{2\sqrt{\gamma_2}} \frac{e^{it\lambda}(2\gamma_2 - \lambda^2)}{\sqrt{4\gamma_2 - \lambda^2}} d\lambda. \tag{4.48}$$

Therefore,

$$\begin{aligned}
Z'(t) &= i\alpha \int_{-\infty}^{\infty} \widehat{\varphi_1}(t)Y(x,t)dt + i\beta \int_{-\infty}^{\infty} \widehat{\varphi_2}(t)\widetilde{Y}(x,t)dt \\
&= -xVZ(x) - \alpha\frac{\kappa_4 xZ(x)}{2\pi\gamma_1^2} \int_{-\infty}^{\infty} \widehat{\varphi_1}(t)\left[\alpha\gamma_1^2 B_{\varphi_1} + \beta\gamma_{12}^2 B_{\varphi_2}\right] \int_{-2\sqrt{\gamma_1}}^{2\sqrt{\gamma_1}} \frac{e^{it\lambda}(2\gamma_1 - \lambda^2)}{\sqrt{4\gamma_1 - \lambda^2}} d\lambda dt \\
&\quad -\beta\frac{\kappa_4 xZ(x)}{2\pi\gamma_2^2} \int_{-\infty}^{\infty} \widehat{\varphi_2}(t)\left[\alpha\gamma_{12}^2 B_{\varphi_1} + \beta\gamma_2^2 B_{\varphi_2}\right] \int_{-2\sqrt{\gamma_2}}^{2\sqrt{\gamma_2}} \frac{e^{it\lambda}(2\gamma_2 - \lambda^2)}{\sqrt{4\gamma_2 - \lambda^2}} d\lambda dt \\
&= -xVZ(x) - \alpha^2\frac{xZ(x)}{2}\gamma_1\gamma_2 B_{\varphi_1}^2 - \alpha\beta xZ(x)\gamma_{12}^2 B_{\varphi_1}B_{\varphi_2} - \beta^2\frac{xZ(x)}{2}\gamma_1\gamma_2 B_{\varphi_2}^2 \\
&= -xVZ(x) - x\kappa_4 Z(x)\left[\alpha^2\gamma_1^2\frac{B_{\varphi_1}^2}{2} + \alpha\beta\gamma_{12}^2 B_{\varphi_1}B_{\varphi_2} + \beta^2\gamma_2^2\frac{B_{\varphi_2}^2}{2}\right] \qquad (4.49)
\end{aligned}$$

where

$$V = \alpha^2\mathbf{Var}(G_1) + 2\alpha\beta\mathbf{Cov}(G_1,G_2) + \beta^2\mathbf{Var}(G_2),$$

and G_1, G_2 are the random variables in Theorem 2.1 with $d = 2$.

Therefore,

$$\begin{aligned}
&\lim_{n\to\infty}\mathbf{Cov}(\mathcal{N}_n^{(1)\circ}[\varphi_1], \mathcal{N}_n^{(2)\circ}[\varphi_2]) = \\
&\quad\mathbf{Cov}(G_1,G_2) + \frac{\gamma_{12}^2\kappa_4}{2\pi^2\gamma_1^2\gamma_2^2} \int_{-2\sqrt{\gamma_1}}^{2\sqrt{\gamma_1}} \varphi_1(\mu)\frac{2\gamma_1 - \mu^2}{\sqrt{4\gamma_1 - \mu^2}}d\mu \int_{-2\sqrt{\gamma_2}}^{2\sqrt{\gamma_2}} \varphi_2(\mu)\frac{2\gamma_2 - \mu^2}{\sqrt{4\gamma_2 - \mu^2}}d\mu. \qquad (4.50)
\end{aligned}$$

By symmetry, for any $1 \le l \le p \le n$,

$$\mathbf{Cov}(\tilde{G}_l, \tilde{G}_p) = \mathbf{Cov}(G_l, G_p) + \frac{\gamma_{lp}^2 \kappa_4}{2\pi^2 \gamma_l^2 \gamma_p^2} \int_{-2\sqrt{\gamma_l}}^{2\sqrt{\gamma_l}} \varphi_l(\lambda) \frac{2\gamma_l - \lambda^2}{\sqrt{4\gamma_l - \lambda^2}} d\lambda \int_{-2\sqrt{\gamma_p}}^{2\sqrt{\gamma_p}} \varphi_p(\mu) \frac{2\gamma_p - \mu^2}{\sqrt{4\gamma_p - \mu^2}} d\mu \tag{4.51}$$

Appendices

A Decoupling Formula

Theorem A.1 (Decoupling Formula). *[?][10] Let ξ be a random variable such that $\mathbb{E}\{|\xi|^{p+2}\} < \infty$ for a certain nonnegative integer p. Then for any function $f : \mathbb{R} \to \mathbb{C}$ of the class C^{p+1} with bounded derivatives $f^{(l)}, l = 1, ..., p+1$, we have*

$$\mathbb{E}\{\xi f(\xi)\} = \sum_{l=0}^{p} \frac{\kappa_{l+1}}{l!} \mathbb{E}\{f^{(l)}(\xi)\} + \varepsilon_p. \tag{A.1}$$

where κ_l denotes the lth cumulant of ξ and the remainder term ε_p admits the bound

$$|\varepsilon_p| \le C_p \mathbb{E}\{|\xi|^{p+2}\} \sup_{t \in \mathbb{R}} f^{(p+1)}(t), \ C_p \le \frac{1 + (3+2p)^{p+2}}{(p+1)!}. \tag{A.2}$$

If ξ is a Gaussian random variable with zero mean,

$$\mathbb{E}\{\xi f(\xi)\} = \mathbb{E}\{\xi^2\} \mathbb{E}\{f'(\xi)\}. \tag{A.3}$$

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